

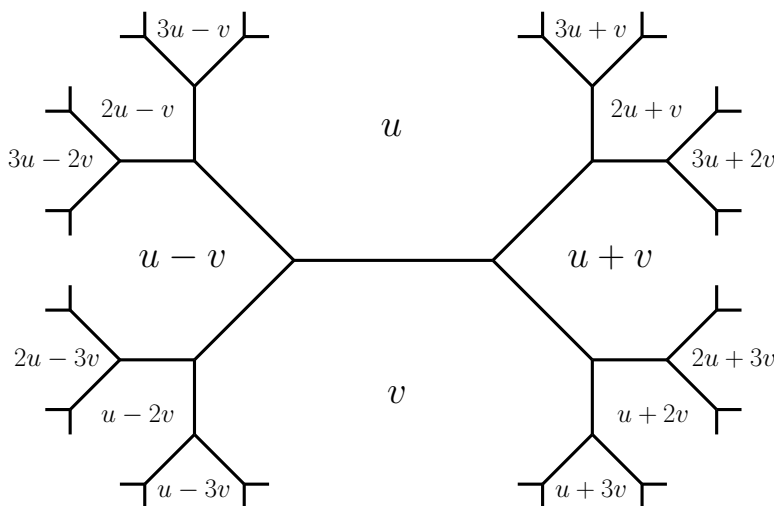
The sensual Apollonian circle packing

KATHERINE E. STANGE

ABSTRACT. The curvatures of the circles in integral Apollonian circle packings, named for Apollonius of Perga (262-190 BC), form an infinite collection of integers whose Diophantine properties have recently seen a surge in interest. Here, we give a new description of Apollonian circle packings built upon the study of the collection of bases of $\mathbb{Z}[i]^2$, inspired by, and intimately related to, the ‘sensual quadratic form’ of Conway.

1. INTRODUCTION

In their delightful monograph entitled *The Sensual Quadratic Form* [8], Conway and Fung draw the following picture of $\mathbb{P}^1(\mathbb{Q})$.



The elements of $\mathbb{P}^1(\mathbb{Q})$ can be viewed as the primitive vectors¹ of \mathbb{Z}^2 , considered up to sign, so that u in the image above represents $\pm u$; such an equivalence class is called a *lax vector*. These lax vectors label an infinite froth of planar regions demarcated by an infinite tree of valence three. More precisely, this branching *topograph* is the graph whose edges correspond to bases of \mathbb{Z}^2 and whose vertices correspond to *superbases*, i.e. triples of lax vectors, any two of which form a basis. The fact that an edge connects

Date: October 23, 2012, Draft #1.

2010 Mathematics Subject Classification. Primary: 52C26, 11E39, Secondary: 11E12, 11E16.

Key words and phrases. Apollonian circle packings, projective linear group, topograph, Hermitian form.

The author’s research has been supported by NSF MSPRF 0802915.

¹vectors which are not integer multiples of any other

two vertices reflects the fact that any basis is contained in exactly two possible superbases. That is, $\{\pm u, \pm v\}$ is contained in $\{\pm u, \pm v, \pm(u+v)\}$ and $\{\pm u, \pm v, \pm(u-v)\}$. One then shows that the graph is a single infinite valence-three tree which can be made planar in such a way that it breaks up the plane into regions, the boundary of each consisting of exactly those edges and vertices containing a single lax vector: we label the region with this vector. In the picture above, all the regions are labelled in terms of one arbitrary choice of basis, u and v .

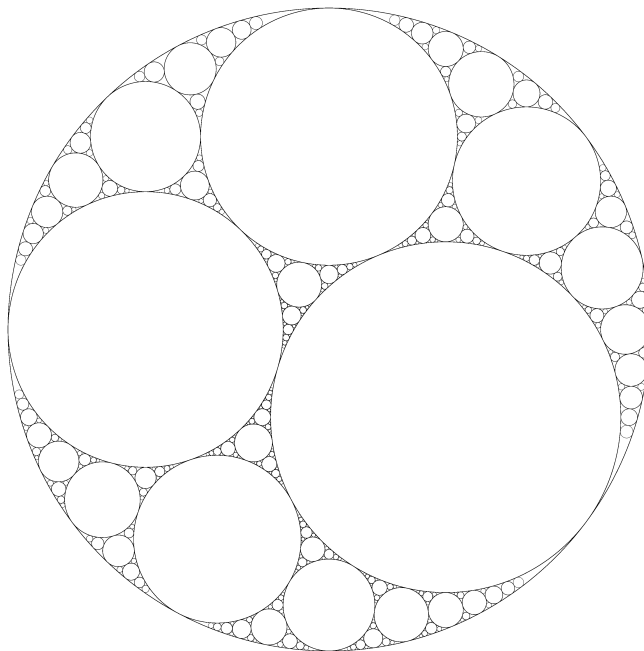
Just as a linear form is determined by its values on a basis, a binary quadratic form is determined by its values on a superbasis. The topograph provides lovely visual evidence of this fact for forms on \mathbb{Z}^2 . Evaluate the quadratic form f on the lax vector labelling each region. The parallelogram law,

$$f(u+v) + f(u-v) = 2f(u) + 2f(v),$$

relates the values on the four regions surrounding an edge of the topograph. Hence knowing the values surrounding any one vertex allows one, iteratively, to deduce every other value in the topograph.

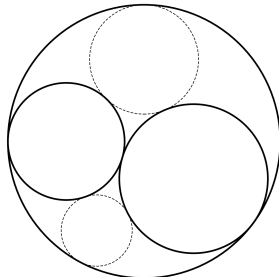
Imagining f as altitude, Conway classifies integral quadratic forms (as indefinite, positive definite, and so on) by their topographical terrain, describing their lakes, wells, weirs and river valleys.

A different, but equally beautiful picture named for geometer Apollonius of Perga (262-190 BC), looks something like this.

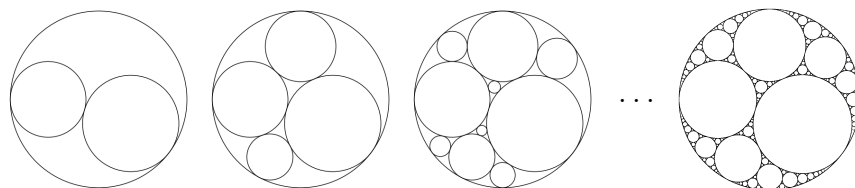


To form such an *Apollonian circle packing*, one starts with three mutually tangent circles of disjoint interiors (sometimes the “interior” of a circle may be defined to be the outside). Given such a triple, there are exactly two

ways to *complete* it to a collection of four mutually tangent circles, again of disjoint interiors, which is called a *Descartes quadruple*. In the following picture, a triple is drawn in solid lines, and the two possible completions are dotted.



To form an Apollonian circle packing, start with any three mutually tangent circles, and add in the missing completions, thereby growing the set of circles. Repeat this process ad infinitum, at each stage adding all the missing completions of all triples in the collection.



The interest for the number theorist lies in the *quadratic Descartes relation* between the curvatures (inverse radii) a, b, c, d of four circles in a Descartes quadruple:

$$2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2. \quad (1)$$

(For circles whose “interior is outside,” curvature is negative.) This relation, which has been traced by Pedoe [18] to Descartes, in a letter to Princess Elisabeth of Bohemia [10, p.49], entails that the two possible curvatures d and d' which complete a triple of curvatures a, b, c to a Descartes quadruple satisfy the *linear Descartes relation*:

$$d + d' = 2(a + b + c). \quad (2)$$

In particular, an Apollonian circle packing seeded with a Descartes quadruple of *integer* curvatures a, b, c, d consists entirely of circles of integer curvature! Such a packing is called *integral*. Furthermore, it is *primitive* if its curvatures have no common factor.

The pheromones exuded by such a mysterious collection of integers have inevitably attracted number theorists, with some remarkable results. See [3, 4, 11, 15, 19] for theorems on the density of the set in the integers, the occurrence of prime curvatures, and much more.

Perhaps the reader has, at this point, noticed a similarity in the stories of Conway and Apollonius. In the first story, each basis (pair of vectors) can be completed to a superbasis (triple of vectors) in exactly two ways,

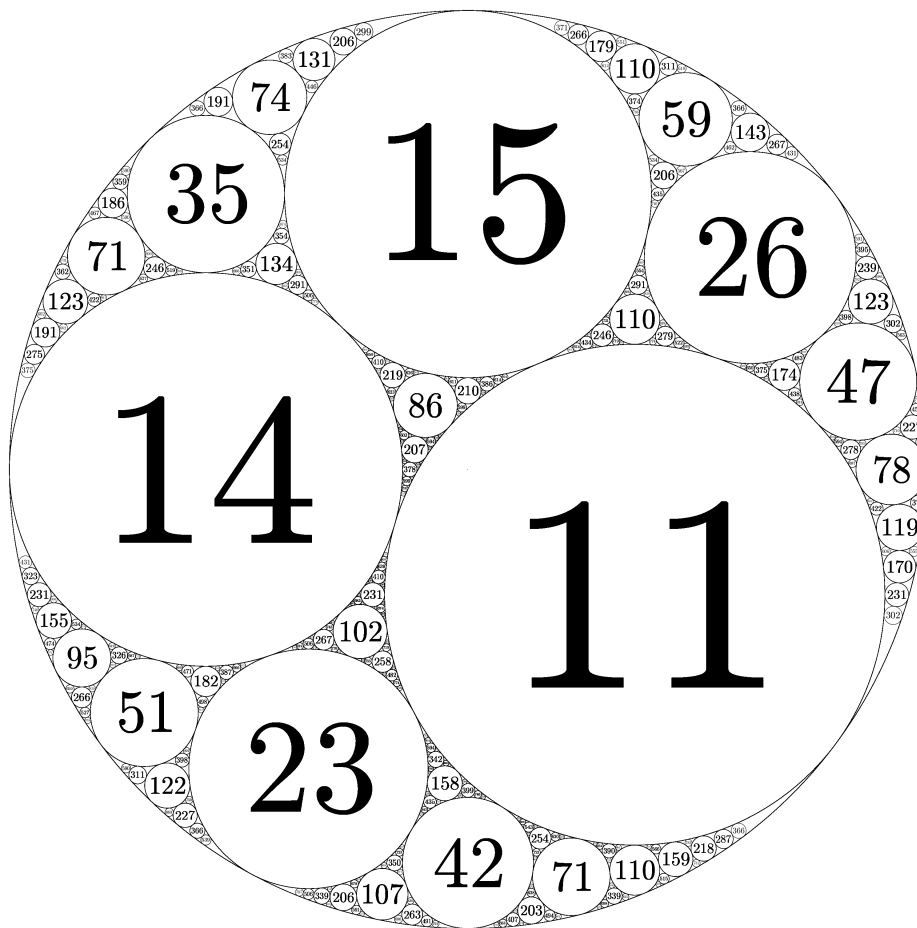


FIGURE 1. A primitive integral Apollonian circle packing in which each circle is labelled with its curvature. The outer circle has curvature -6 .

and the values of f on the two new possibilities sum to twice the sum on the original pair:

$$f(u+v) + f(u-v) = 2(f(u) + f(v)).$$

In the second story, each triple of tangent circles can be completed to a Descartes quadruple in exactly two ways, and the curvatures of the two new possibilities sum to twice the sum on the original triple:

$$d + d' = 2(a + b + c).$$

Just as Conway labels the topograph with a quadratic form whose values satisfy the parallelograph law, it is natural to ask if there is a corresponding ‘Apollonian topograph,’ which we can label with some type of form to obtain the curvatures of an Apollonian circle packing. We aim to describe just such an object: where Conway’s topograph was essentially a picture of

$\mathbb{P}^1(\mathbb{Q})$, we describe an object intimately related to $\mathbb{P}^1(\mathbb{Q}(i))$. Where Conway labelled the topograph with values of a binary quadratic form, we consider Hermitian forms.

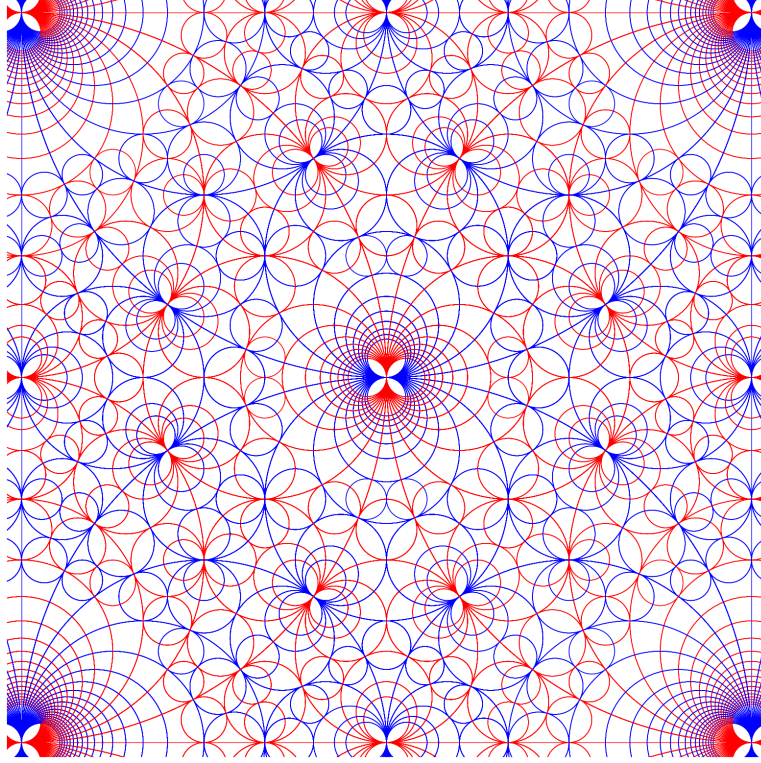


FIGURE 2. A portion of an Apollonian superpacking (red), together with its dual (blue). The image shows those circles of curvature less than 20 within a square with side-length 2, centred on $1 + i$. See Section 13.

The ‘Apollonian topograph’ described in this paper is a forest. It consists of infinitely many trees of valence four, each of which is, in a concrete sense, an Apollonian circle packing: its vertices correspond to Descartes quadruples, and two quadruples are adjacent when they share three circles in common (Theorem 4.1). The Apollonian forest, in a different form called the ‘Apollonian city’ (formed of infinitely many ‘palaces’), can be labelled with the imaginary part of any Hermitian form. For a particular choice of form, we obtain the curvatures of the circles in the corresponding packings; it is also possible to obtain the centres in this manner (Theorem 4.2).

With this perspective comes a natural way to associate a full rank sublattice of \mathbb{Z}^2 to each circle in any Apollonian circle packing, and there is a corresponding ‘lattice Descartes rule’ (Theorem 4.5) which describes the lattices of the two completing circles for a triple. To each circle we also discover an associated equivalence class of quadratic forms, whose values,

translated, give the curvatures of its tangent circles (Theorem 4.6). That such forms existed was known [3, 11]; here we obtain a precise description of these forms in terms of circle lattices, which, in particular, describes the form of a circle in terms of the forms of its neighbours (Theorem 4.7).

Taking the entire forest as one object, we obtain a union of packings called an *Apollonian superpacking*, first introduced in [13], and shown in Figure 2. The superpacking contains copies of every possible primitive Apollonian circle packing. Through its connection to the Apollonian forest, the superpacking can be viewed as a beautiful picture of the collection of full-rank sublattices of \mathbb{Z}^2 , a picture of $\mathrm{PGL}_2(\mathbb{Z}[i])/\mathrm{PSL}_2(\mathbb{Z})$, or, from another perspective, a piece of a picture of $\mathbb{P}^1(\mathbb{Q}(i))$. Another piece of the picture of $\mathbb{P}^1(\mathbb{Q}(i))$ inspired by Conway’s topograph was studied by Bestvina and Savin in [2] (see Section 17).

Acknowledgements. The author would like to thank Lionel Levine for drawing her attention to Apollonian circle packings, and would especially like to thank David Wilson for sharing data he had collected associating circles and lattices in the context of abelian sandpiles. It was an examination of his data that led to this study. For more on this connection, see [17]. The author owes a debt of gratitude to the invaluable and detailed suggestions provided by Jeffrey Lagarias, Andrew Granville, Jonathan Wise and an anonymous referee on an earlier draft. The images in this paper were produced using IPE [6] and Sage Mathematics Software [20].

2. THE APOLLONIAN CITY

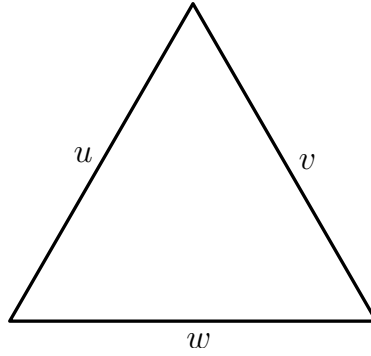
Our central object is the *Apollonian city* and its labelling by *lax vertex lattices*. After defining these in two brief sections, we can then state the Main Bijection relating the Apollonian city to Apollonian circle packings. Later, we will see how this labelled object is the analogue of Conway’s topograph.

Extending terminology from Conway and Fung’s monograph, a *lax vector* is an equivalence class $\{u, -u, iu, -iu\}$ for some primitive² $u \in \mathbb{Z}[i]^2$. A *superbasis*³ is a collection of three lax vectors $\{u, v, w\}$, any two of which form a basis for $\mathbb{Z}[i]^2$. This entails that $\epsilon_1 u + \epsilon_2 v + \epsilon_3 w = 0$, for some appropriate choice of units ϵ_i .

The 2-cells of the Apollonian city will consist of the collection of all superbases of $\mathbb{Z}[i]^2$, in the form of triangles whose sides are labelled with the superbasis elements u , v and w :

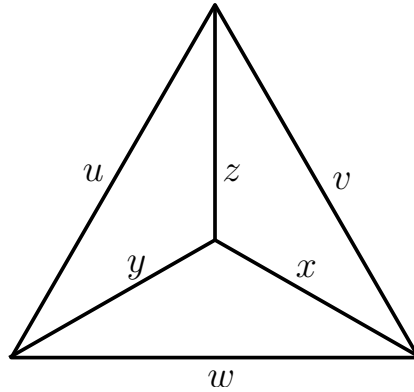
²i.e. not a Gaussian integer multiple of another vector of $\mathbb{Z}[i]^2$.

³If we consider $\mathbb{P}^1(\mathbb{Q}(i))$ as $\mathbb{P}^1(\mathbb{Z}[i])$ over $\mathrm{Spec} \mathbb{Z}[i]$, then a superbasis can be defined as three distinct $\mathbb{Z}[i]$ -points of $\mathbb{P}^1(\mathbb{Z}[i])$.



We will call the 2-cells *walls*.

An *ultrabasis* is a collection of six lax vectors $\{u, v, w, x, y, z\}$ such that the four subsets $\{u, v, w\}$, $\{u, y, z\}$, $\{v, x, z\}$ and $\{w, x, y\}$ are superbases (as it turns out, if a set of six lax vectors is an ultrabasis, it is an ultrabasis in exactly two ways; we consider these distinct). Whenever the union of four superbases forms an ultrabasis, the sides of the corresponding triangular 2-cells are identified according to their labels, and the Apollonian city includes a tetrahedral 3-cell whose boundary consists of the four triangles:



We will call the 3-cells *chambers*.

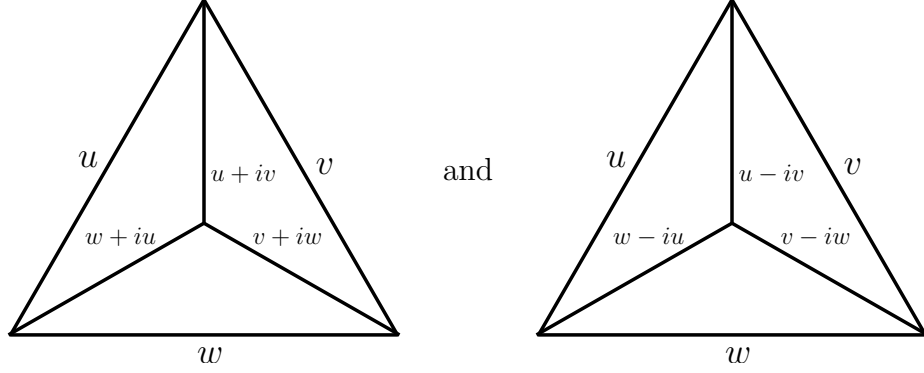
This information is sufficient to define the Apollonian city as a 3-dimensional cell complex whose edges are labelled with lax vectors: simply glue the collection of all 2-cells according to their formation into superbasis 3-cells. According to the construction, a given lax vector may appear as a label on more than one edge, but a given superbasis or ultrabasis has a unique appearance as a 2-cell or 3-cell respectively.

Each component of the Apollonian city is called a *palace*.

We will see that each chamber is adjacent to exactly four others, sharing one wall with each; the valence-four graph whose vertices are chambers connected according to adjacency is called the *Apollonian Forest* (described in more detail in Section 5); we will see that it consists of infinitely many trees.

3. THE VERTICES OF THE APOLLONIAN CITY

Suppose we have a superbasis represented by vectors u, v, w with the convention that $u + v + w = 0$ (i.e., representatives u, v and w are chosen so that no unit multiples are required to cause the sum to vanish). Then there are exactly two ways to form an ultrabasis containing this superbasis⁴:

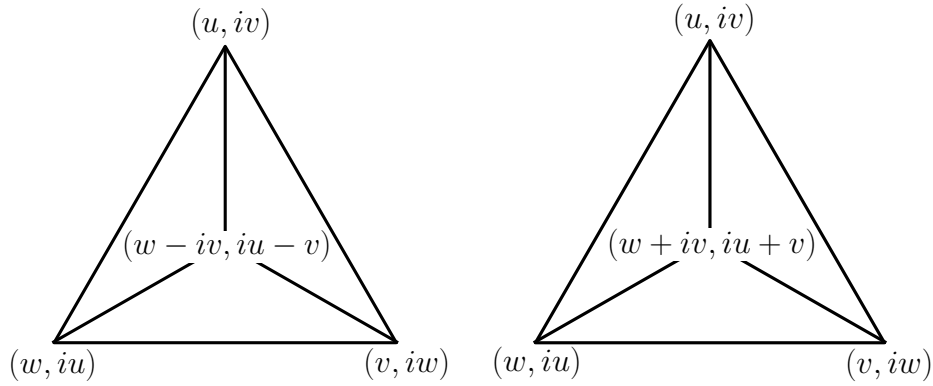


For example, the leftmost triangle is a superbasis because

$$u - i(u + iv) + (w + iu) = u + v + w = 0.$$

These two ultrabases share the face u, v, w in the Apollonian city. Any ultrabasis is of the form of one of these two *standard ultrabases*, for some u, v, w ; and any pair of chambers sharing a wall in the city is of the form of this pair.

We will now label each vertex of the standard ultrabases above with an ordered pair of vectors, according to the following picture:



Definition 3.1. Considering $\mathbb{Z}[i]^2$ as a \mathbb{Z} -module of rank four, let L be an oriented \mathbb{Z} -submodule of rank two generated by an oriented $\mathbb{Z}[i]$ -basis for $\mathbb{Z}[i]^2$. The lattice L is called a *vertex lattice* and the pair $\{L, iL\}$ is called a *lax vertex lattice*.

⁴Note that the three edges touching a single vertex in an ultrabasis also form a superbasis (see Section 17 for more about this ‘duality’).

For example, the lattice $\{a(-i, -2 - i) + b(-1, -1 + i) : a, b \in \mathbb{Z}\}$ is a vertex lattice, since the matrix $\begin{pmatrix} -i & -1 \\ -2 - i & -1 + i \end{pmatrix}$ has determinant -1 . This vertex lattice, together with the vertex lattice $\{a(1, 1 - 2i) + b(-i, -1 - i) : a, b \in \mathbb{Z}\}$ form a lax vertex lattice.

We will attach to each vertex of the Apollonian city the lax vertex lattice generated by the labels in the picture above (i.e. label each ultrabasis as a standard ultrabasis for some u, v, w , and then attach labels as shown).

Proposition 3.2. *The lax vertex lattice attached to a vertex in the Apollonian city is well-defined.*

The proof can be found in Section 7.

4. FROM A CITY TO A PACKING: MAIN RESULTS

The space $\mathbb{P}^1(\mathbb{C})$ is defined as the collection of non-zero vectors in \mathbb{C}^2 modulo multiplication by non-zero scalars. We identify $\mathbb{P}^1(\mathbb{C})$ with points of $\mathbb{C} \cup \{\infty\}$ by the map $(a, b) \mapsto a/b$. The subset $\mathbb{P}^1(\mathbb{Q}(i))$ consists of those equivalence classes of vectors containing a vector in $\mathbb{Q}(i)^2$; one may identify this with the collection of primitive vectors of $\mathbb{Z}[i]^2$ up to sign (i.e. lax vectors).

Möbius transformations act on $\mathbb{P}^1(\mathbb{C})$ via the matrix action of $\mathrm{GL}_2(\mathbb{C})$ on vectors in \mathbb{C}^2 . Since matrices which are scalar multiples have the same action, we can consider instead the action of equivalence classes of matrices up to non-zero scalar multiple; this is the group $\mathrm{PGL}_2(\mathbb{C})$. The group $\mathrm{PGL}_2(\mathbb{Z}[i])$ is taken to be the subgroup of $\mathrm{PGL}_2(\mathbb{C})$ formed of equivalence classes containing a matrix with entries in $\mathbb{Z}[i]$ and determinant a unit in $\mathbb{Z}[i]$. Those with such a representative whose determinant is ± 1 form the smaller subgroup $\mathrm{PSL}_2(\mathbb{Z}[i])$. Finally, within this group, some may have a representative with entries in \mathbb{Z} and determinant 1; this is $\mathrm{PSL}_2(\mathbb{Z})$. We obtain a chain of subgroups:

$$\mathrm{PGL}_2(\mathbb{C}) > \mathrm{PGL}_2(\mathbb{Z}[i]) > \mathrm{PSL}_2(\mathbb{Z}[i]) > \mathrm{PSL}_2(\mathbb{Z}).$$

In what follows we will frequently refer to an equivalence class of $\mathrm{PGL}_2(\mathbb{C})$ by a matrix representative. The group $\mathrm{PGL}_2(\mathbb{Z}[i])$ takes elements of $\mathbb{P}^1(\mathbb{Q}(i))$ to itself, so it can be considered to act on the lax primitive vectors of $\mathbb{Z}[i]^2$, which will sometimes be identified with elements of $\mathbb{Q}(i) \cup \{\infty\}$ via the map $(a, b) \mapsto a/b$ above.

As described in the introduction, an Apollonian circle packing is a collection of infinitely many circles in $\mathbb{P}^1(\mathbb{C}) \cong \mathbb{C} \cup \{\infty\}$ (where lines are considered circles passing through ∞) with disjoint interiors, formed from an initial triple of three mutually tangent circles by an iterative process. At each stage, for every triple of mutually tangent circles already included, one adds to the collection the two circles which complete that triple to a

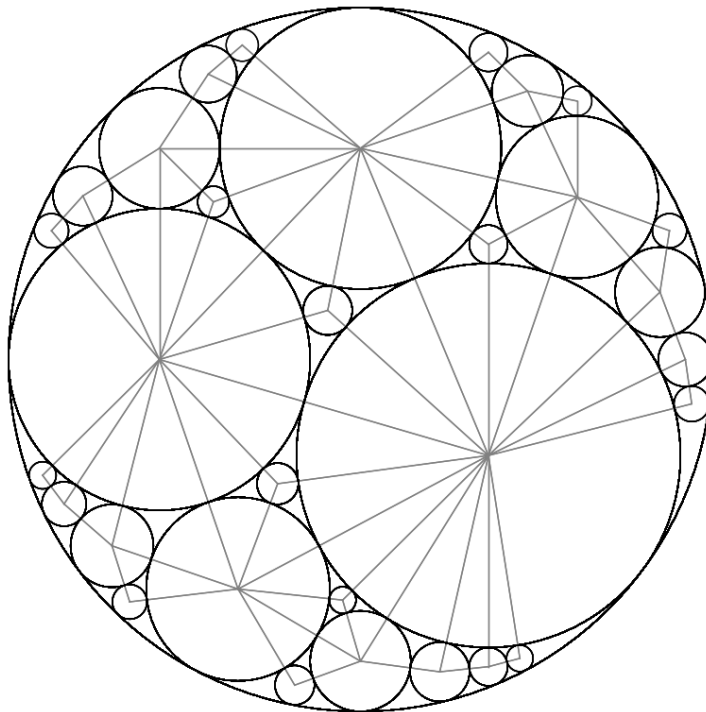


FIGURE 3. An Apollonian circle packing (in black) with the Apollonian city (in grey) shown as an overlay; the vertices of the city are located at the centre of the corresponding circles. (Only circles and vertices of the Apollonian city with curvatures below a small bound are shown; the vertex corresponding to the outermost circle is omitted.)

Descartes quadruple (we will call these the *completing circles*). In the limit, we obtain a well-defined infinite collection of circles.

Such a packing, or a subset of one, is called *integral* if all of its curvatures are integral, and *strongly integral* if, in addition, the centre of each circle, multiplied by the curvature, is a Gaussian integer. It is called *primitive* if the collection of curvatures has no non-trivial common factor.

Each circle comes with a designated interior; equivalently, we can consider the circle oriented so that the interior is the region to the right as one travels around the circle in the direction of orientation.

The central theorem (the ‘Main Bijection’) describes the relationship between the Apollonian city and Apollonian circle packings: we show that the Apollonian city comes in infinitely many connected components, or *palaces*, the vertices of a particular palace being in bijection with the circles in a particular primitive strongly integral Apollonian circle packing, where adjacent vertices correspond to tangent circles. Placing each vertex of the city at the centre of the corresponding circle of the packing, we obtain pictures

like Figure 3. As a consequence of the Main Bijection, the Apollonian city is an example of a class of graphs named *Apollonian networks* after this connection [1].

Theorem 4.1 (Summary of Theorems 8.1 and 12.1). *We have a $\mathrm{PGL}_2(\mathbb{Z}[i])$ -equivariant bijection (the ‘Main Bijection’) between the following sets:*

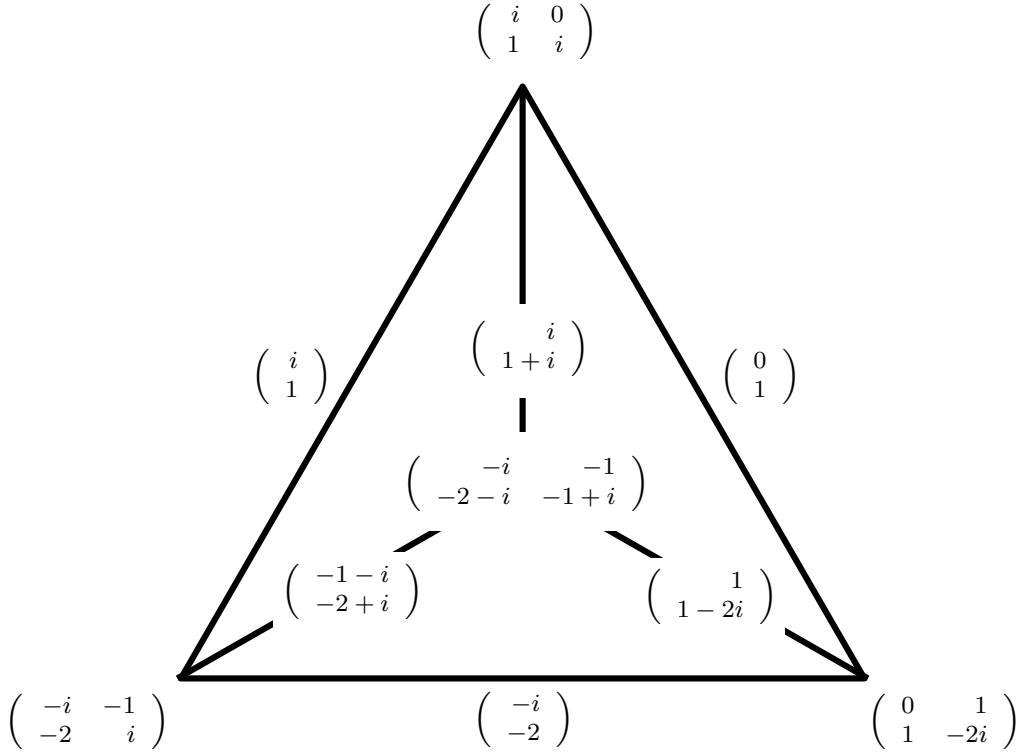
$$\begin{array}{c}
 \left\{ \begin{array}{l} \text{oriented circles and lines in } \mathbb{C} \\ \text{which are images of the real line} \\ \text{under Möbius transformations} \end{array} \right\} \\
 \text{in } \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \mathrm{PGL}_2(\mathbb{Z}[i]) \\
 \updownarrow \\
 \mathrm{PGL}_2(\mathbb{Z}[i]) / \mathrm{PSL}_2(\mathbb{Z}) \\
 \updownarrow \\
 \{\text{lax vertex lattices}\} \\
 \updownarrow \\
 \{\text{vertices of the Apollonian city}\}
 \end{array}$$

Under the bijection, circles which are tangent within an Apollonian circle packing correspond to vertices connected by an edge in the Apollonian city, and these properties are preserved by the action. In particular, the vertices of a chamber in the city correspond to a primitive strongly integral Descartes quadruple of circles.

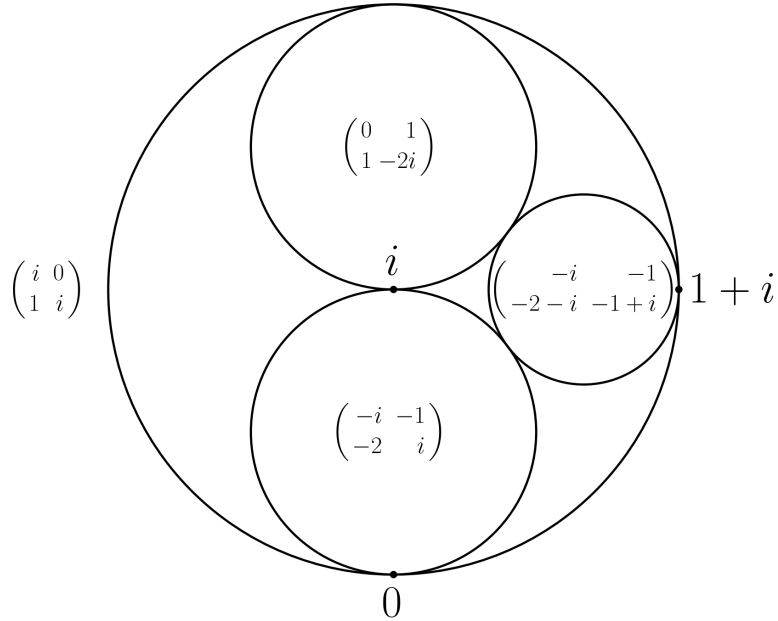
Furthermore, the Apollonian city consists of infinitely many components, each of which corresponds to a unique primitive strongly integral Apollonian circle packing. Finally, every primitive Apollonian circle packing appears, up to rigid motions.

The bijection is mediated through the lax vertex lattice: its basis vectors form the columns of the corresponding element of $\mathrm{PGL}_2(\mathbb{Z}[i])$.

It may be informative to carry along an example of this bijection. Choosing $u = (i, 1)$ and $v = (0, 1)$, by convention we set $w = (-i, -2)$ (so that $u + v + w = 0$). We obtain the following ultrabasis tetrahedron, with vertex lattices shown (as matrices whose column vectors generate the lattice):



The corresponding images of \mathbb{R} in \mathcal{S}_1 form a Descartes quadruple:



The circles of \mathcal{S}_1 fill the complex plane densely. Figure 2 shows those circles that have curvature less than 20. The red circles, no two of which cross transversely, correspond to the subgroup $\text{PSL}_2(\mathbb{Z}[i])$ of $\text{PGL}_2(\mathbb{Z}[i])$; the blue circles correspond to its nontrivial coset.

Just as Conway's topograph could be labelled with various quadratic forms, we will label the Apollonian city with the imaginary part of a Hermitian form at each vertex (evaluated via the vertex lattice). These values satisfy the linear Descartes rule, and for a special choice of Hermitian form, we obtain the curvatures, and the products of the centres and curvatures of the circles.

Theorem 4.2 (Summary of Propositions 10.2 and 11.1). *Let $H : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{R}$ be the imaginary part of a Hermitian form. Then H takes a well-defined value on a lax vertex lattice, and hence on a vertex of the Apollonian city, where this value is obtained by evaluating H on any \mathbb{Z} -basis of the lattice. Consider the values A , B and C of H on the vertices of the common wall between a pair of adjacent chambers in the Apollonian city. Let D and D' be the values at the other two vertices of the chambers. Then*

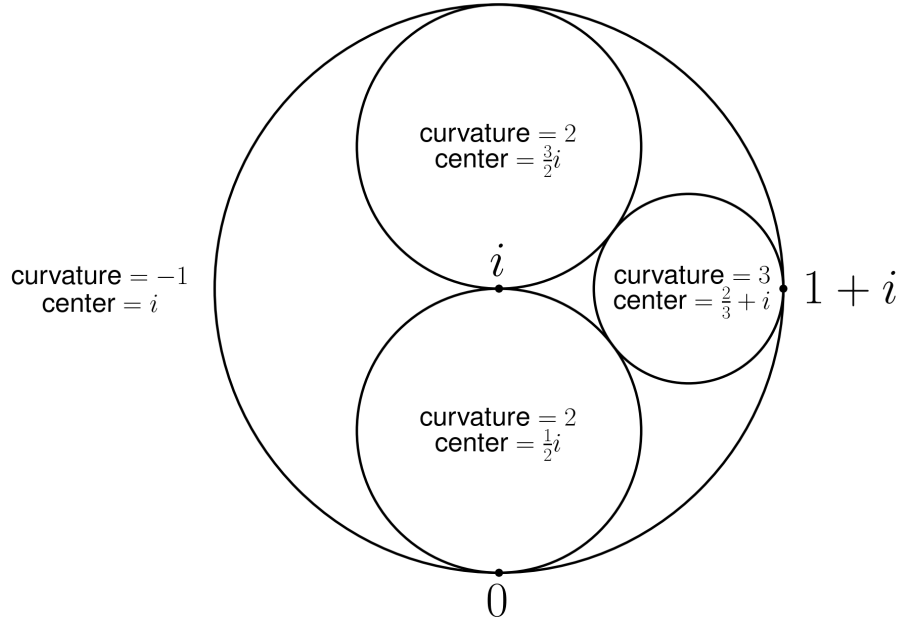
$$D + D' = 2(A + B + C).$$

In particular, consider the following imaginary parts of Hermitian forms:

$$\begin{aligned} H_1((\alpha, \beta), (\gamma, \delta)) &= \text{Im}(\beta\bar{\delta}), \\ H_2((\alpha, \beta), (\gamma, \delta)) &= \text{Im}(\beta\bar{\gamma} + \alpha\bar{\delta}), \\ H_3((\alpha, \beta), (\gamma, \delta)) &= \text{Im}(i\beta\bar{\gamma} - i\alpha\bar{\delta}). \end{aligned}$$

The value of H_1 is the curvature of the corresponding circle under the Main Bijection, and $H_2 + iH_3$ is the curvature times the center of the circle.

Continuing the example quadruple above, we find the centres and curvatures of the Descartes quadruple as follows:



Because H_1 of Theorem 4.2, whose values give curvatures, depends only on the second coordinates of its vector inputs, one may choose to restrict attention at a vertex of the Apollonian city from its vertex lattice to the \mathbb{Z} -module generated just by the second coordinates of the vertex lattice.

Definition 4.3. Considering $\mathbb{Z}[i]$ as a \mathbb{Z} -module of rank two, let L be a full-rank \mathbb{Z} -submodule. The lattice L is called a *circle lattice*. If L has basis (a, b) and (c, d) , then the pair $\{L, L'\}$ where L' has basis $(b, -a)$, $(d, -c)$, is called a *lax circle lattice*.

For example, the vertex lattice $\{a(-i, -2 - i) + b(-1, -1 + i) : a, b \in \mathbb{Z}\}$, corresponds to the circle lattice $\{a(-2, -1) + b(-1, 1) : a, b \in \mathbb{Z}\}$.

Thus we associate to each vertex of the Apollonian city a lax circle lattice. This is no longer a bijection; each lax circle lattice may appear many times.

Proposition 4.4 (See Proposition 15.1). *The determinant of a circle lattice is the negative of the curvature of the associated circle.*

Just as there is a Descartes rule for curvatures, we can now record a ‘Descartes rule for circle lattices’.

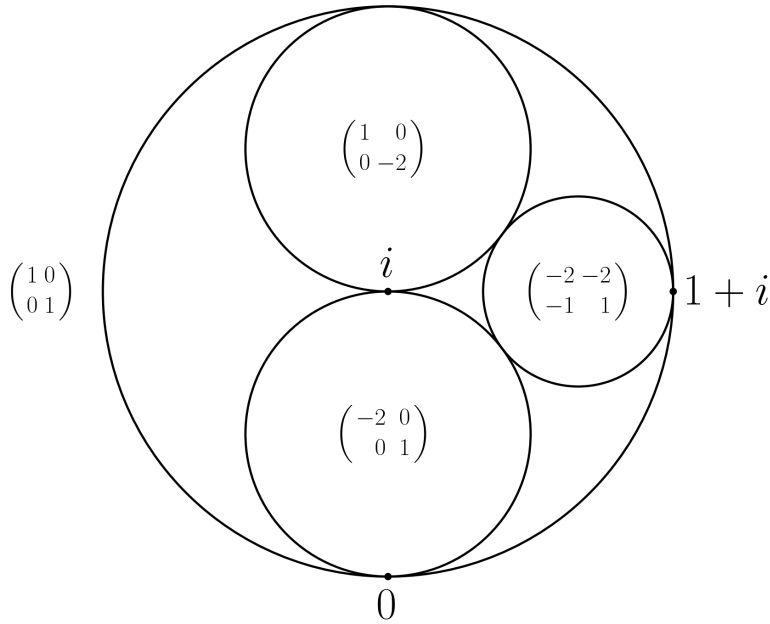
Theorem 4.5 (Theorem 15.2). *Consider three mutually tangent circles in an Apollonian circle packing. For some integers a, b, c, d , they have the following circle lattices (represented by matrices whose column vectors form a basis):*

$$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \quad B = \begin{pmatrix} d & b - c \\ -c & -a - d \end{pmatrix}, \quad C = \begin{pmatrix} -a - d & -b \\ -b + c & a \end{pmatrix}.$$

The two completing circles have the following circle lattices:

$$D_1 = \begin{pmatrix} -b - d & c + a + d \\ a + c & d + b - c \end{pmatrix}, \quad D_2 = \begin{pmatrix} c - a & d - b + c \\ d - b & -c + a + d \end{pmatrix}.$$

Figure 4 shows the circle lattices associated to the circles in an example Apollonian circle packing. In the example we have been following, the circle lattices are as follows:



The relationship to Conway's topograph and the Apollonian city extends beyond an analogy: one obtains infinitely many copies of the topograph (in a slightly different form) within the Apollonian city.

Theorem 4.6 (Introductory form of Theorem 14.1). *Fix a vertex of the Apollonian city with vertex lattice L . Then the adjacent vertices may be put in bijection with the primitive elements $x \in L$ in such a way that any two such vertices are connected by an edge of the Apollonian city if and only if their corresponding $x, x' \in L$ form a basis for L . Furthermore, for H the imaginary part of a Hermitian form, the values of H at the vertices are the values of a translated binary quadratic form on L .*

If H is the curvature form H_1 of Theorem 4.2, then the quadratic form described in this theorem is given by the circle lattice of the fixed vertex in a simple way: if M is a matrix representation of the circle lattice, then $M^T M$ is the matrix of the quadratic form. As a consequence, the form has discriminant negative four times the square of the curvature of the circle.

Theorem 4.7 (Corollary 16.2). *We have an explicitly computable bijection:*

$$\begin{array}{c} \{\text{lax circle lattices}\} \\ \updownarrow \\ \left\{ \begin{array}{l} \text{equivalence classes of primitive positive definite} \\ \text{integral binary quadratic forms} \\ \text{of discriminant negative four times a square} \end{array} \right\} \end{array}$$

To any circle in an Apollonian circle packing are associated a lax circle lattice and hence a quadratic form; the values of the form, translated by

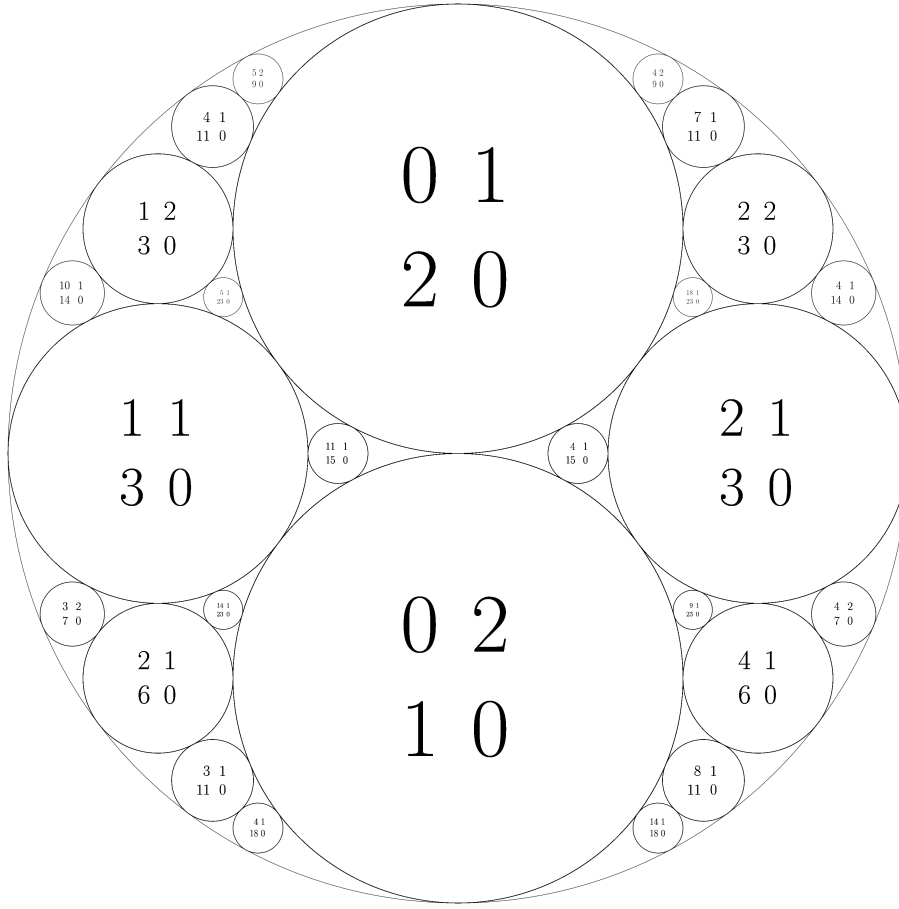


FIGURE 4. An Apollonian circle packing with circle lattices.
(Only circles with curvatures below a small bound are shown.)

the determinant of the lax circle lattice, are exactly the curvatures of all the circles in the Apollonian circle packing which are tangent to the given circle.

For example, fixing the lower circle of curvature 2 in Figure 4 gives rise to the quadratic form $X^2 + 4Y^2$. The curvatures of the tangent circles are:

$$-1, 2, 3, 6, 11, 15, 18, \dots$$

which list, when translated by 2 (the curvature of the fixed circle), are exactly the values

$$1, 4, 5, 8, 13, 17, 20, \dots$$

of the quadratic form.

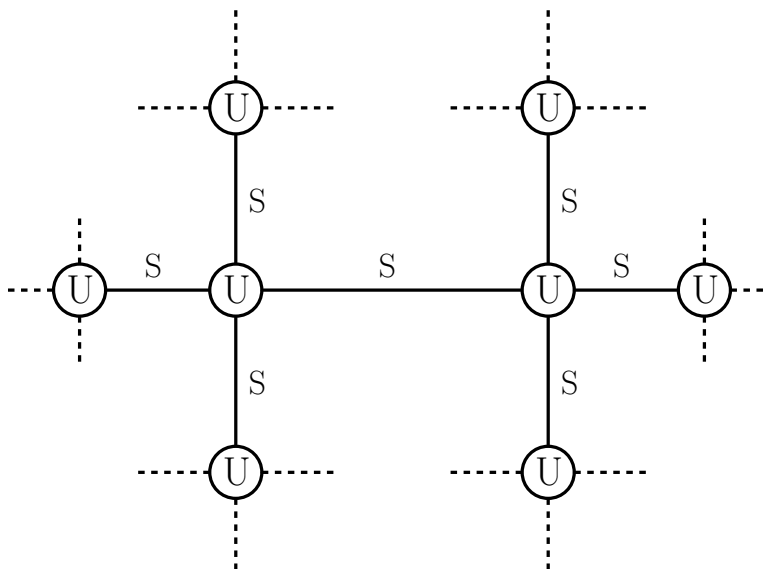
This recovers a relationship between Apollonian circle packings and quadratic forms that has already been studied; see for example [12]. In fact,

it is one of the main ingredients in the proof of several major Diophantine results concerning the curvatures of an Apollonian circle packing [3, 4]. Through Theorems 4.5 and 4.7, we obtain a simple new rule for relating the quadratic forms of tangent circles.

5. THE APOLLONIAN FOREST

The *Apollonian forest*, shortly to be defined as a type of dual to the Apollonian city, is at times more convenient, and it is more clearly an analogue to Conway's topograph.

The forest is a graph whose vertices and edges are the collection of ultrabases and superbases of $\mathbb{Z}[i]^2$, respectively. Their adjacency is defined by the property of an ultrabasis containing a superbasis as one of its four defining superbases. Hence we obtain a valence-four graph. A small piece of the Apollonian forest looks like this, where \textcircled{U} represents an ultrabasis vertex, and $\text{---}\overset{S}{\text{---}}$ represents a superbasis edge:

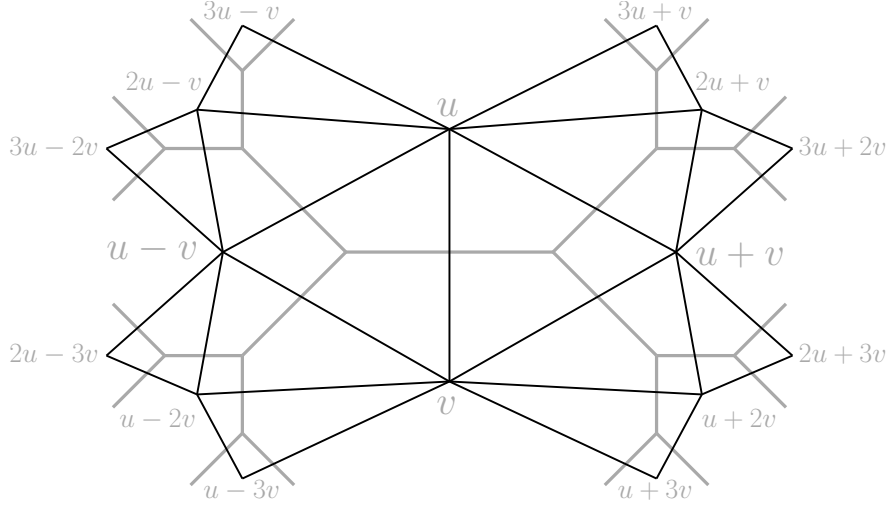


Superimposing the forest on the city, the forest's vertices can be considered to lie in the center of the corresponding chambers of the city, and its edges to pierce the walls of the chambers (this is the 'duality' referred to above).

Once we have established the Main Bijection, each component of the Apollonian forest will be in bijection with an Apollonian circle packing; the tree is in fact the graph whose vertices are the Descartes quadruples of the packing, and whose edges connect pairs of Descartes quadruples which share a common triple of circles.

The Apollonian forest is the true analogue of Conway's topograph. Suppose we form a 2-dimensional cell complex based on the topograph, where each face gives rise to a vertex, and faces which touch become vertices

connected by an edge. Each vertex of the topograph gives rise to a 2-dimensional triangular face. In this new picture, each vertex is labelled with a vector up to sign, each edge denotes a basis, and each face (triangle) denotes a superbasis. This new object, shown in black against the grey topograph in the following picture, is the analogue to the Apollonian city.



It is possible, although we do not prove it here, to embed the Apollonian forest in three-dimensional space in such a way that it breaks the space up into regions, each corresponding to a lax vertex lattice, i.e. a vertex of the Apollonian city.

6. THE APOLLONIAN FOREST CONSISTS OF TREES

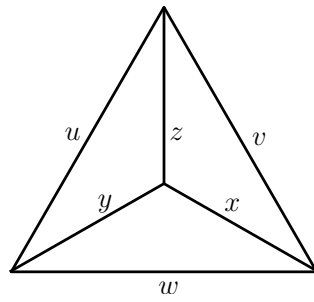
The following will be needed to prove Proposition 3.2, among others.

Proposition 6.1. *Every component of the Apollonian forest is a tree.*

Proof. Consider the form $f : \mathbb{Z}[i]^2 \rightarrow \mathbb{Z}$ given by

$$f(a + bi, c + di) = a^2 + b^2.$$

This induces a labelling on the vertices (ultrabases) of the forest as follows: label an ultrabasis of the form



with the integer

$$f(u) + f(v) + f(w) + f(x) + f(y) + f(z),$$

which we will call the *height* of the ultrabasis. This is well defined since $f(u) = f(-u) = f(iu) = f(-iu)$.

We will shortly show that at any vertex, at least three of the four adjacent vertices have a strictly greater height. If this is the case, then, starting out at any vertex and travelling upward, we must travel upward at every subsequent step, unless we double back. From this, one concludes that there are no non-trivial loops.

We consider one of the standard ultrabases and its four neighbours (in particular, $u + v + w = 0$), shown in Figure 5.

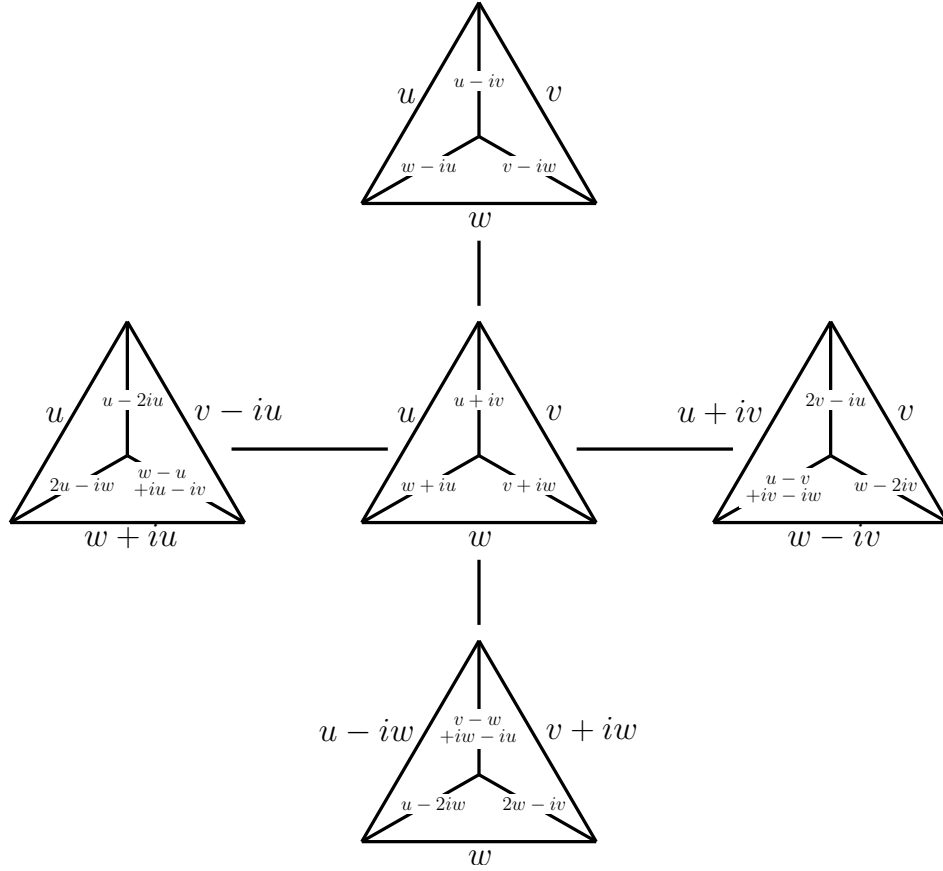


FIGURE 5. A portion of an Apollonian tree.

Suppose that u has first coordinate $a+bi$, v has first coordinate $c+di$, and w has first coordinate $(-a-c) + (-b-d)i$. Then the height of the central ultrabasis minus the heights, respectively, of each of the four neighbours is, clockwise from top:

$$\begin{aligned} & -12(ad - bc), & & -12(c^2 + d^2 - ad + bc), \\ & -12((a+c)^2 + (b+d)^2 - ad + bc), & & -12(a^2 + b^2 - ad + bc). \end{aligned}$$

Since u, v form a basis for $\mathbb{Z}[i]$, their first coordinates, $a + bi$ and $c + di$, must be coprime, hence $ad - bc \neq 0$. If the first of the four quantities listed above is non-negative, then all the others are negative. In other words, if one of the four neighbours has an equal or lower height than the central ultrabasis, then the other three are higher. The proof for the other standard ultrabasis is much the same. Without loss of generality, this proves the general case. \square

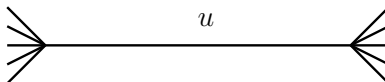
We will see later that the Apollonian forest consists of infinitely many trees.

7. LAX VERTEX LATTICES

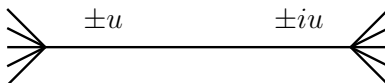
This section contains the proof of Proposition 3.2 showing that the vertex lattices associated to a vertex of the Apollonian city are well-defined; as a consequence of the proof, we also discover that each lax vertex lattice appears exactly once in the Apollonian city.

Proof of Proposition 3.2. As a means of proof, we will give an equivalent definition of the labelling of the vertices which is clearly well-defined. This involves adding a little extra structure to the labelling of the Apollonian city.

Given an edge in the city labelled with u , representing an equivalence class of vectors modulo *units*, we instead label each end of the edge (i.e. we label the edge together with a choice of one of its two vertices; we place the label closer to the chosen vertex) with an equivalence class of vectors modulo *sign*. The two sign-equivalence classes have as their union the original unit-equivalence class, so that the edge

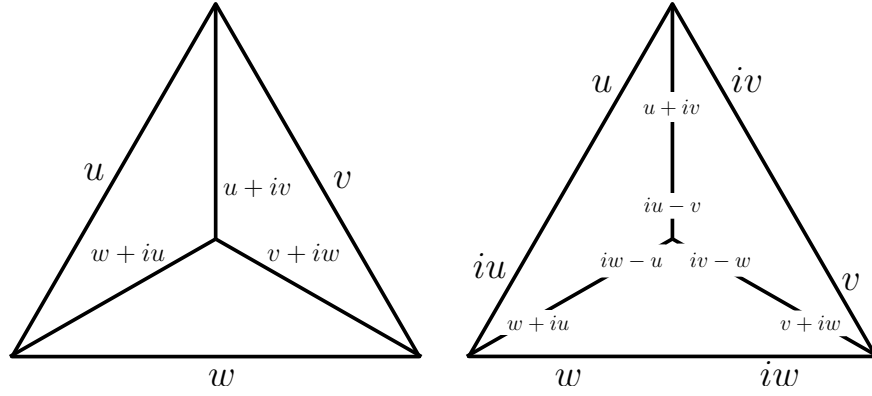


becomes



There are, of course, two ways to do this (differing by multiplication by i).

This *splitting of labels* can be performed for every edge of a tetrahedral chamber in such a way that surrounding a vertex, the three nearby labels x , y and z satisfy $x + y + z = 0$ for some choice of *signs* (instead of unit multiples). Up to multiplication of every label by i , there is a unique way to do this. For example, here is a standard ultrabasis tetrahedron before and after splitting (signs are suppressed).



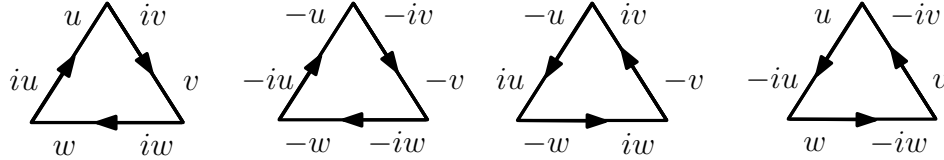
Since the forest has no loops (Proposition 6.1), this shows that, once a single label has been split, there is a unique way to split labels on all the ultrabases in a single palace simultaneously and consistently. Multiplying every label simultaneously by i allows us to move from one of the two possible global splittings to the other. (Choosing one of these two global splittings is an arbitrary choice we will make for the moment; this discomfiture will be resolved shortly.)

Now each vertex in the Apollonian city has many proximal labels on its adjacent edges, but any three arising from one tetrahedron satisfy $x+y+z=0$ for some choice of signs. This implies that, having selected two such proximal labels (from a single tetrahedron among those sharing that vertex), say u and iv , all other proximal labels are \mathbb{Z} -linear combinations of u and iv . This defines a rank two $\mathbb{Z}[i]^2$, where the latter is considered as a free \mathbb{Z} -module of rank 4; this will be the vertex lattice. The submodule is generated by two primitive vectors forming a $\mathbb{Z}[i]$ -basis for $\mathbb{Z}[i]^2$.

All that remains is to explain how to determine the orientation of the vertex lattice. Any triangular side which touches the vertex has proximal labels x and y and distal labels ix and iy (up to sign) on the two edges connected to that vertex. Choose the signs of the labels on the entire triangle so that two things hold:

- (1) if multiplication by $-i$ is considered to orient an edge, the triangle is a cycle,
- (2) the ‘forward’ labels on the three edges sum to zero.

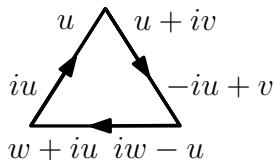
For example, if $u+v+w=0$, then there are exactly four ways to do this:



At this point the labels represent vectors, not vectors up to unit or sign.

Now, put an order on the basis generating the submodule of our chosen vertex according to the orientation of the triangle. In this case, the ordered

basis at the upper vertex is (u, iv) , $(-u, -iv)$, $(iv, -u)$ or $(-iv, u)$, for the four pictures, respectively, but they all represent the *same* oriented lattice. It remains to verify that if we do this with another wall of the same chamber (under the same global splitting), we obtain the same oriented lattice at a common vertex. It suffices to check this for one other wall of a standard ultrabasis⁵:



At this point we have now defined a vertex lattice attached to the specified vertex of the Apollonian city. However, we made one arbitrary binary choice in the process, resulting in two possible global labellings of the vertices of each palace, differing by multiplication by i .

Hence, what we have actually defined, without making an arbitrary choice, is a lax vertex lattice.

Finally, the lax vertex lattices associated to the standard ultrabases are exactly those defined in Section 3. \square

In some circumstances, for a single palace, it will be convenient to make the arbitrary binary choice described in the proof above, and talk about the (non-lax) vertex lattice at a vertex.

We can summarize the ‘meaning’ of the various pieces of the Apollonian city or forest in a table.

Apollonian forest	Apollonian city	meaning (up to unit)
	vertex	lax vertex lattice
	edge	primitive vector of $\mathbb{Z}[i]^2$
edge	triangle (wall)	superbasis of $\mathbb{Z}[i]^2$
vertex	tetrahedron (chamber)	ultrabasis of $\mathbb{Z}[i]^2$

Proposition 7.1. *With the lax vertex lattice labelling described above, each lax vertex lattice appears exactly once in the Apollonian city.*

Proof. For every basis u, v of $\mathbb{Z}[i]^2$, there are exactly four possible lax vertex lattices (exemplified by bases (u, v) , (u, iv) , $(u, -v)$ and $(u, -iv)$). All four appear somewhere in the Apollonian city. For example, the lattice generated by u, iv appears on one vertex of the superbasis $u, v, -u - v$, while the lattice generated by u, v appears on one vertex of the superbasis $u, -iv, -u + iv$. In fact, each lax vertex lattice appears *only once*, since the \mathbb{Z} -bases (u, v) are

⁵Note that this sign-refinement of the split labelling and orienting of edges cannot be performed in a single consistent way simultaneously throughout a palace. But it is not necessary to do so.

in bijection with the superbasis walls $u, iv, -u - iv$ which share that vertex (this follows from the proof of Proposition 3.2), and hence the lax vertex lattice uniquely determines the collection of superbasis walls it touches. \square

8. THE MAIN BIJECTION

Theorem 8.1. *Consider the following sets with actions by $\mathrm{PGL}_2(\mathbb{Z}[i])$:*

(1)

$$\mathcal{S}_1 = \left\{ \begin{array}{l} \text{oriented circles and lines in } \mathbb{C} \\ \text{which are images of the real line} \\ \text{under Möbius transformations} \\ \left(\begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} M \text{ for } M \in \mathrm{PGL}_2(\mathbb{Z}[i]) \end{array} \right\}$$

with action by left multiplication on M .

(2) $\mathcal{S}_2 = \mathrm{PGL}_2(\mathbb{Z}[i]) / \mathrm{PSL}_2(\mathbb{Z})$ *with action by left multiplication.*

(3) $\mathcal{S}_3 = \{\text{lax vertex lattices}\}$ *with action on the lax vertex lattice.*

(4) $\mathcal{S}_4 = \{\text{vertices of the Apollonian city}\}$ *with action on the lax vertex lattice labelling the vertex.*

These four sets with action are in $\mathrm{PGL}_2(\mathbb{Z}[i])$ -equivariant bijection. In particular, $M \in \mathcal{S}_2$ corresponds to

(1) *the image of $\begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} M$ in \mathcal{S}_1 ,*

(2) *the lax vertex lattice generated by the column vectors of M in \mathcal{S}_3 ,*

(3) *the vertex labelled with the lax vertex lattice generated by the column vectors of M in \mathcal{S}_4 .*

Proof. Represent a lax vertex lattice by a matrix M whose column vectors form an ordered \mathbb{Z} -basis for the lattice. Matrices whose column vectors form a basis for the same lax vertex lattice must be considered equivalent. In particular,

(1) M is equivalent to MN for $N \in \mathrm{PSL}_2(\mathbb{Z})$,

(2) M is equivalent to iM , $-M$ and $-iM$.

Hence, a lax vertex lattice is in bijection with an element of

$$\mathrm{PGL}_2(\mathbb{Z}[i]) / \mathrm{PSL}_2(\mathbb{Z})$$

(here the quotient is by right multiplication).

The group of Möbius transformations $\mathrm{PGL}_2(\mathbb{Z}[i])$ takes circles to circles in $\mathbb{P}^1(\mathbb{C})$ (where as usual, straight lines are considered circles passing through ∞). Since $\mathrm{PSL}_2(\mathbb{Z})$ is the group of Möbius transformations in $\mathrm{PGL}_2(\mathbb{Z}[i])$ that fix the real line, preserving its orientation, to specify an element $M \in \mathrm{PGL}_2(\mathbb{Z}[i]) / \mathrm{PSL}_2(\mathbb{Z})$ is to specify an oriented circle which is the image of the real line (oriented to the right, say) under M .

Clearly, these bijections preserve the actions of $\mathrm{PGL}_2(\mathbb{Z}[i])$. Finally, we have already seen that lax vertex lattices are in bijection with vertices of the Apollonian city (Proposition 7.1). We have defined the action of $\mathrm{PGL}_2(\mathbb{Z}[i])$

on the vertices by its action on the vertex lattices, so the action is preserved by the bijection. \square

9. THE ACTION OF $\mathrm{PGL}_2(\mathbb{Z}[i])$ ON THE APOLLONIAN CITY

Proposition 9.1. *The action of $\mathrm{PGL}_2(\mathbb{Z}[i])$ in Theorem 8.1 preserves adjacencies in the Apollonian city, or equivalently, the Apollonian forest. The Apollonian forest consists of infinitely many trees, and the subgroup taking any one tree to itself is a conjugate of*

$$\Gamma = \left\langle \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \right\rangle < \mathrm{PGL}_2(\mathbb{Z}[i]).$$

Proof. The group $\mathrm{PGL}_2(\mathbb{Z}[i])$ acts on $\mathbb{P}^1(\mathbb{Q}(i))$ in such a way that a superbasis is taken to another superbasis, and an ultrabasis is taken to another ultrabasis. This gives an action of $\mathrm{PGL}_2(\mathbb{Z}[i])$ on the Apollonian forest, or equivalently, on the Apollonian city, preserving adjacencies. This action is conjugate to the action in Theorem 8.1 on the vertices of the Apollonian city, because the action of M on a basis u, v corresponds to the action of $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}^{-1} M \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ on the lax vertex lattice with ordered basis u, iv . The remainder of the proof will discuss this action in lieu of the action in Theorem 8.1. It will suffice that the actions are conjugate.

Consider a superbasis $u + v + w = 0$. The subgroup fixing this edge of the forest, in terms of basis u, v , is generated by the two elements

$$F_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad F_3 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

The matrix F_i is of order i , and this group of order 6 corresponds to the permutations of the elements u, v and w . Although these elements fix the edge, they do not fix the entire forest. Instead, they correspond to *flipping* the edge (swapping its two vertices), and *rotating* the edge (acting cyclically on the three adjacent edges at each of its vertices).

One edge has six adjacent edges. Representative elements moving the edge to each of these adjacent edges are

$$\begin{aligned} M &= \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}, F_3 M F_3^2 = \begin{pmatrix} 1+i & -i \\ i & 1-i \end{pmatrix}, F_3^2 M F_3 = \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix}, \\ M^{-1} &= \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix}, F_3 M^{-1} F_3^2 = \begin{pmatrix} 1-i & i \\ -i & 1+i \end{pmatrix}, F_3^2 M^{-1} F_3 = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Therefore, the tree in the forest containing a given superbasis consists of the orbit of that superbasis under the group Γ generated by all these matrices. Since $F_2 = F_3^2 M F_3 M^{-1} F_3^2 M F_3$, we have

$$\Gamma = \langle M, F_3 \rangle.$$

The first generator is parabolic and the second elliptic of order 3. This subgroup has infinite index in $\mathrm{PGL}_2(\mathbb{Z}[i])$ [7, Table 4], hence the Apollonian forest has infinitely many connected components. By Proposition 6.1, they are all trees. \square

The action of a certain group called the *Apollonian group* on the collection of ordered, oriented Descartes quadruples of curvatures has been a principle tool in the study of Apollonian circle packings [12, 14]. The Main Bijection shows that each tree in the Apollonian forest is associated to an Apollonian circle packing in such a way that the vertices correspond to Descartes quadruples and edges indicate quadruples which share a common triple of circles. The action described above is different than the action of the Apollonian group. For example, the quadruples are unordered, and the action above does not take every vertex to an adjacent vertex.

10. LABELLING THE APOLLONIAN CITY WITH A HERMITIAN FORM

In order to establish the rest of the properties of the bijection in Theorem 4.1, we must establish several results relating to Hermitian forms. We will describe a method to display the values of a Hermitian form on both the edges and vertices of the Apollonian city, in analogy to the values of a quadratic form displayed on Conway's topograph.

A *Hermitian form* is a pairing

$$h : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}$$

satisfying for any $u, v, w \in \mathbb{C}^n$ and $a \in \mathbb{C}$,

- (1) $h(au + v, w) = ah(u, w) + h(v, w)$
- (2) $h(u, av + w) = \bar{a}h(u, v) + h(u, w)$
- (3) $h(u, v) = \overline{h(v, u)}$

We will study H , the imaginary part of h (i.e. the coefficient of i). Then H has the following important property:

$$H(au + bv, cu + dv) = (ad - bc)H(u, v) \text{ for } a, b, c, d \in \mathbb{R}. \quad (\text{H1})$$

(Note: since $ih(u, v) = h(u, -iv)$, we may have equivalently chosen to study the collection of real parts of h ; in fact, h is determined by H .)

From now on, we will assume h is *integer valued* on $\mathbb{Z}[i]^2 \times \mathbb{Z}[i]^2$, i.e. it takes values in $\mathbb{Z}[i]$.

We can evaluate H at a vertex of the Apollonian city by applying H to a basis of the lax vertex lattice. Properties (H1) and conjugate linearity together guarantee that the H -value is well-defined. We can also evaluate H on an edge of the Apollonian city by applying the map $H(ix, x)$ to the edge label x (also well-defined by conjugate linearity).

As a demonstration, here is the usual superbasis triangle labelled with u , v and w (where, as always, $u + v + w = 0$). A split labelling and orientation

(as in the proof of Proposition 3.2) are also shown, to demonstrate that they align nicely with the H -labelling.

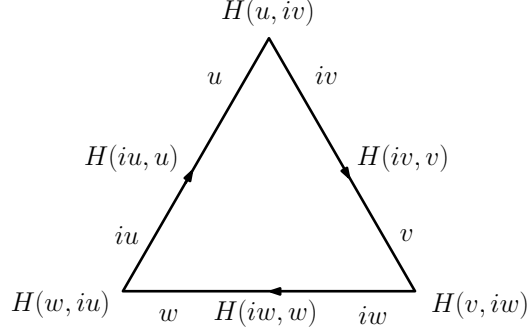


Figure 6 shows the Hermitian labels on the first and second standard ultrabasis tetrahedra.

Lemma 10.1. *If $u + v + w = 0$, then the following quantities are real:*

- (1) $h(u, iv) + h(v, iw) - h(iv, v)$
- (2) $h(w - iv, iu - v) + h(w + iv, iu + v) - h(iv, v) - h(iw, w) - h(iu, u)$

Proof. The proof is by direct calculation. It is convenient to note that for a Hermitian form, the following quantities are real:

$$h(u, u), \quad h(u, v) + h(v, u), \quad i(h(u, v) - h(v, u)).$$

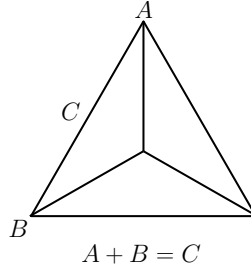
Note also that $h(ix, iy) = h(x, y)$ and that $h(ix, y) = -h(x, iy)$ for all x, y . \square

As a consequence, we discover that the H -labels on the pair of standard ultrabasis tetrahedra have the following relationships:

Proposition 10.2. *With the H -labelling described above, the following hold:*

- *the label on an edge is the sum of the labels at its endpoints, i.e.*

$$H(u, iv) + H(v, iw) = H(iv, v) \quad (\text{H2})$$



- *if the two standard ultrabases are considered together, the labels on their common edges add up to the sum of the labels of their non-common vertices, i.e.*

$$H(w - iv, iu - v) + H(w + iv, iu + v) = H(iv, v) + H(iw, w) + H(iu, u) \quad (\text{H3})$$

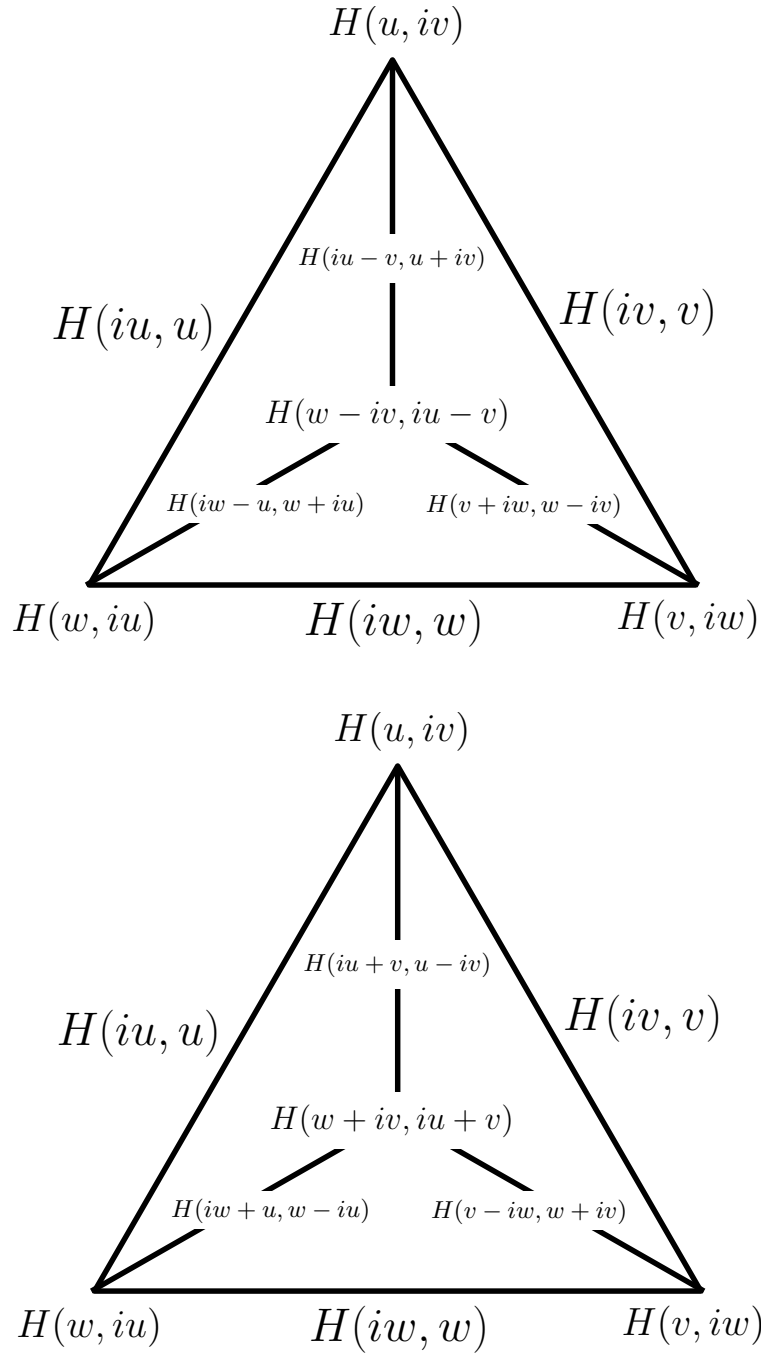
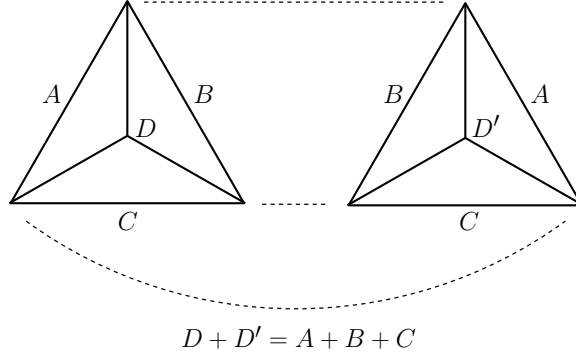
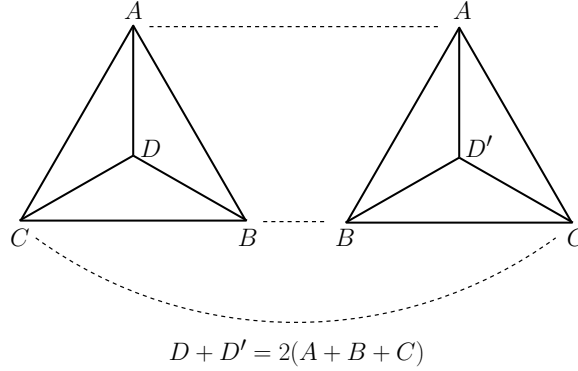


FIGURE 6. The first and second standard ultrabasis tetrahedra, labelled with the imaginary part of a Hermitian form. Note that the labels on the second are the same as the first, except that sums become differences and differences become sums.



- if the two standard ultrabases are considered together, the labels on their common vertices, doubled, add up to the sum of the labels of their non-common vertices, i.e.

$$H(w-iv, iu-v) + H(w+iv, iu+v) = 2(H(u, iv) + H(v, iw) + H(w, iu)) \quad (\text{H4})$$



By Proposition 7.1, the collection of Hermitian values

$$\{H(u, v) : u, v \text{ form a } \mathbb{Z}[i]\text{-basis for } \mathbb{Z}[i]^2\}$$

is now displayed on the vertices of the Apollonian city. Property (H4) can be considered the linear Descartes rule for Hermitian forms.

11. THE CURVATURE AND CENTRE OF A CIRCLE IN $\mathbb{P}^1(\mathbb{C})$

By *oriented curvature* of an oriented circle, we mean a quantity that is positive if the circle is oriented clockwise, negative if oriented counterclockwise, and is equal to the usual curvature in absolute value.

Proposition 11.1. *Let*

$$M = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}.$$

Then the oriented curvature of the oriented circle which is the image of the real axis (oriented to the right) under M , is given by

$$\text{Im}(\beta\bar{\delta}).$$

Furthermore, the product of the curvature and centre of the circle is given by

$$i(\gamma\bar{\beta} - \alpha\bar{\delta}) = \operatorname{Im}(\beta\bar{\gamma} + \alpha\bar{\delta}) + i \operatorname{Im}(i\beta\bar{\gamma} - i\alpha\bar{\delta})$$

Furthermore, this value has even real part and odd imaginary part whenever $M \in \operatorname{PSL}_2(\mathbb{Z}[i])$; it has odd real part and even imaginary part whenever $M \in \operatorname{PGL}_2(\mathbb{Z}[i]) \setminus \operatorname{PSL}_2(\mathbb{Z}[i])$.

Finally, the circles in \mathcal{S}_1 of Theorem 8.1 have integer curvature, and the product of curvature and centre is a Gaussian integer.

Proof of Proposition 11.1. Define functions on elements $N = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$ of $\operatorname{PGL}_2(\mathbb{Z}[i])$ by

$$\begin{aligned} H_1(N) &= \operatorname{Im}(2\beta\bar{\delta}), \\ H_2(N) &= \operatorname{Im}(\beta\bar{\gamma} + \alpha\bar{\delta}), \\ H_3(N) &= \operatorname{Im}(i\beta\bar{\gamma} - i\alpha\bar{\delta}). \end{aligned}$$

Then each H_i is the imaginary part of a Hermitian form on the column vectors of the matrix N (note that H_1 is twice the form in the statement of the Proposition). We will show that the image of the real line under N has curvature $H_1(N)$ and center \times curvature $H_2(N) + iH_3(N)$, from which the statement of the Proposition follows (since the image of M is the circle obtained from the image of N by dilating the complex plane by a factor of two).

Each H_i , and the image circle itself, is invariant under replacing N by any other transformation $N' = NP$ for some $P \in \operatorname{PSL}_2(\mathbb{R})$ (see property (H1) of the imaginary part of a Hermitian form in Section 10). If we change the orientation of the circle, replacing N with $N \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, then the values of the H_i change sign.

Keeping track of the possible change of orientation and sign, we can replace N by a transformation of the following form (using an appropriate $P \in \operatorname{PSL}_2(\mathbb{R})$ and possibly $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ as above), whose image is the same circle:

$$N' = \begin{pmatrix} \frac{k+b}{\sqrt{2k}} & i\frac{k-b}{\sqrt{2k}} \\ \frac{1}{\sqrt{2k}} & \frac{-i}{\sqrt{2k}} \end{pmatrix} \in \operatorname{PGL}_2(\mathbb{C}), \quad b \in \mathbb{Q}(i), k \in \mathbb{Q}.$$

The image of the real axis under this N' is the clockwise circle with centre b and radius k . To verify these statements, compose the transformation $\begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ 1/\sqrt{2} & -i/\sqrt{2} \end{pmatrix}$ taking \mathbb{R} (oriented to the right) to the clockwise unit circle, with a dilation $\begin{pmatrix} \sqrt{k} & 0 \\ 0 & 1/\sqrt{k} \end{pmatrix}$ and a translation $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, all given by elements of $\operatorname{PGL}_2(\mathbb{C})$.

For this N' , $H_1(N') = 1/k$, and $b/k = H_2(N') + iH_3(N')$. Note that

$$i(\gamma\bar{\beta} - \alpha\bar{\delta}) = \text{Im}(\beta\bar{\gamma} + \alpha\bar{\delta}) + i \text{Im}(i\beta\bar{\gamma} - i\alpha\bar{\delta}).$$

Finally,

$$\alpha\delta - \gamma\beta - (\alpha\bar{\delta} - \gamma\bar{\beta}) = \alpha(\delta - \bar{\delta}) - \gamma(\beta - \bar{\beta}) \equiv 0 \pmod{2}$$

Since $\alpha\delta - \gamma\beta$ is a unit, this means the real and imaginary parts of the centre-times-curvature $i(\gamma\bar{\beta} - \alpha\bar{\delta})$ consist of one even, and one odd, integer. In particular, if N is in $\text{PSL}_2(\mathbb{Z}[i])$, then the real part is even, and the imaginary part is odd. If N is in the nontrivial coset of $\text{PSL}_2(\mathbb{Z}[i])$ in $\text{PGL}_2(\mathbb{Z}[i])$, then the opposite is true. \square

12. DESCARTES QUADRUPLES CORRESPOND TO APOLLONIAN CHAMBERS

The goal of this section is to prove the following theorem, which will complete the proof of Theorem 4.1.

Theorem 12.1. *Under the bijection of Theorem 8.1, primitive Descartes configurations of circles in \mathcal{S}_1 are in bijection with chambers of the Apollonian city. Furthermore, the palaces of the Apollonian city are in bijection with the collection of primitive Apollonian circle packings in \mathcal{S}_1 . Finally, up to rigid motions, every primitive Descartes configuration of circles appears in \mathcal{S}_1 , as does every primitive Apollonian circle packing.*

We require a sequence of propositions and lemmas.

Proposition 12.2. *The circles of \mathcal{S}_1 in bijection with two adjacent vertices of the Apollonian city (as in Theorem 8.1) are tangent.*

Proof. The lax vertex lattices at two adjacent vertices can be given in the form

$$\langle u, iv \rangle \text{ and } \langle iv, u + v \rangle.$$

Therefore, the corresponding circles both contain the point iv (considering a primitive vector of $\mathbb{Z}[i]$ as an element of $\mathbb{P}^1(\mathbb{C})$).

Suppose the circles have two distinct points in common. The new point can be viewed as a non-zero primitive vector x in the \mathbb{Z} -span of u and iv . Then either x or ix is in the span of iv and $u + v$. We will show that neither is possible.

If x is in the span of iv and $u + v$, then the two (non-lax) vertex lattices $\langle u, iv \rangle$ and $\langle iv, u + v \rangle$ have two independent vectors iv and x in common. Hence they have the sum in common, and the two circles contain at least three distinct points in common, and are therefore the same circle (up to orientation). Hence the two lax vertex lattices are equal up to orientation. But then $v = niv$ for $n \in \mathbb{Z}$, which is not possible.

So it must be that it is ix , and not x , that is in the span of iv and $u + v$. But then x is a common vector in $\langle u, iv \rangle$ and $\langle -v, iu + iv \rangle$. This implies the

existence of a \mathbb{Z} -linear combination of u, v, iu, iv . By the $\mathbb{Z}[i]$ -independence of u and v , the coefficients of this combination must vanish; and this implies $x = 0$, which is contrary to our assumption.

Therefore the two circles have exactly one point in common: they are tangent. \square

Proposition 12.3. *The circles of \mathcal{S}_1 in bijection with the vertices of a chamber of the Apollonian city form a primitive strongly integral Descartes quadruple. Furthermore, every primitive Descartes quadruple appears in \mathcal{S}_1 , up to a rigid motion.*

The proof relies on a lemma.

Lemma 12.4. *If $A, B, C, D \in \mathbb{Z}$ have no common factor, $A, C \geq 0$, and $AC = B^2 + D^2$, then there are $u, v \in \mathbb{Z}[i]$ with $\gcd(u, v) = 1$ such that*

$$A = u\bar{u}, \quad B = \frac{u\bar{v} + \bar{u}v}{2}, \quad C = v\bar{v}, \quad D = \frac{u\bar{v} - \bar{u}v}{2i}. \quad (3)$$

Conversely, if $u, v \in \mathbb{Z}[i]$ are coprime, then A, B, C, D given by (3) are integers with no common factor.

Proof. First, we show that there is a bijection

$$\mathbb{P}^1(\mathbb{Q}(i)) \rightarrow X = \{[A, B, C, D] : AC = B^2 + D^2\} \subset \mathbb{P}^3(\mathbb{Q})$$

given by

$$[u, v] \mapsto \left[u\bar{u}, \frac{u\bar{v} + \bar{u}v}{2}, v\bar{v}, \frac{u\bar{v} - \bar{u}v}{2i} \right].$$

In fact, a brief calculation verifies that the image of the map falls in X . An inverse map is given by

$$[A, B, C, D] \mapsto [A, B - Di] = [B + Di, C].$$

(At least one of these is always defined, and they are equal when both defined since $AC = B^2 + D^2$.)

Now assume that $xu + yv = 1$, for some $x, y \in \mathbb{Z}[i]$. Then

$$x\bar{x}u\bar{u} + \bar{x}y\bar{u}v + x\bar{y}u\bar{v} + y\bar{y}v\bar{v} = 1,$$

or equivalently,

$$x\bar{x}u\bar{u} + (x\bar{y} + \bar{x}y) \left(\frac{u\bar{v} + \bar{u}v}{2} \right) + i(x\bar{y} - \bar{x}y) \left(\frac{u\bar{v} - \bar{u}v}{2i} \right) + y\bar{y}v\bar{v} = 1.$$

Hence if $u, v \in \mathbb{Z}[i]$ are coprime, then A, B, C, D have no common factor. Therefore if we start with A, B, C, D without a common factor, $A, C \geq 0$, $AC = B^2 + D^2$, and we choose some $u, v \in \mathbb{Z}[i]$ which are coprime such that $[u, v] \mapsto [A, B, C, D]$, then (3) must hold exactly. \square

Proof of Proposition 12.3. Consider a chamber of the Apollonian city. Proposition 11.1 implies that the corresponding circles are strongly integral. From Proposition 12.2, the circles form a Descartes quadruple. Suppose

$$u = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad v = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}, \quad w = -u - v.$$

where u and v form a basis for $\mathbb{Z}[i]^2$.

Let $B \begin{pmatrix} \epsilon & \eta \\ \zeta & \theta \end{pmatrix} = \text{Im}(\zeta \bar{\theta})$ give the curvature of the circle which is the image of \mathbb{R} under $\begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \epsilon & \eta \\ \zeta & \theta \end{pmatrix}$, as in Proposition 11.1. Write $\beta = a + bi$ and $\delta = d - ci$ respectively.

One checks that then the curvatures corresponding the standard ultra-basis vertices for u, v, w are given by

$$b_1 = -(ad - bc) = B(u, iv), \tag{4}$$

$$b_2 = c^2 + d^2 + (ad - bc) = B(w, iu), \tag{5}$$

$$b_3 = (a + c)^2 + (b + d)^2 + (ad - bc) = B(v, iw), \tag{6}$$

$$b_4 = a^2 + b^2 + (ad - bc) = B(w - iv, iu - v). \tag{7}$$

It is apparent that any common factor of all the b_i must divide the norms of $a + bi$, $c + di$ and $(a + c) + (b + d)i$, which in turn implies that these three Gaussian integers have a common factor. Since u, iv form a basis for $\mathbb{Z}[i]^2$, this cannot occur. Hence the b_i are coprime, and the quadruple is primitive.

Now consider curvatures $b_1, b_2, b_3, b_4 \in \mathbb{Z}$ satisfying the Descartes relation (1), and having no common factor. Without loss of generality assume that $b_1 + b_4, b_2 + b_4 > 0$. Rearranging (1) gives

$$\frac{1}{4} (b_1 + b_2 - b_3 + b_4)^2 + b_4^2 = (b_1 + b_4)(b_2 + b_4).$$

Note that (1) also entails that $b_1 + b_2 - b_3 + b_4 \equiv 0 \pmod{2}$. By primitivity, the integers $b_1 + b_4$, $b_2 + b_4$, b_4 and $(b_1 + b_2 - b_3 + b_4)/2$ have no common factor. Therefore we are in the case of Lemma 12.4, and there are some $a, b, c, d \in \mathbb{Z}$ such that

$$(a - bi)(-c - di) = \frac{b_1 + b_2 - b_3 + b_4}{2} + b_4 i, \tag{8}$$

$$N(a - bi) = b_1 + b_4, \tag{9}$$

$$N(-c - di) = b_2 + b_4, \tag{10}$$

such that $a + bi$ and $c + di$ are coprime.

We may choose u to have second coordinate $a + bi$ and v to have second coordinate $d - ci$, while the first coordinates may be chosen so that u and v

form a basis for $\mathbb{Z}[i]^2$. This is possible because $a + bi$ and $d - ci$ are coprime. Let $w = -u - v$. Then, from (8)–(10),

$$\begin{aligned} \frac{b_1 + b_2 - b_3 + b_4}{2} &= -(ac + bd), \quad b_4 = -(ad - bc) \\ b_1 + b_4 &= a^2 + b^2, \quad b_2 + b_4 = c^2 + d^2. \end{aligned}$$

and so equations (4)–(7) hold, which demonstrates that we have found a chamber giving the necessary curvatures. Finally, any two Descartes quadruples with the same set of oriented curvatures must be related by a rigid motion. \square

The following lemma is a variation on Lemma 12.4.

Lemma 12.5. *Suppose that $A, B, C, D \in \mathbb{Z}[i]$ are such that $AC = B^2 + D^2$ and both A and C are divisible by 2. Then there are $a, b, c, d \in \mathbb{Z}[i]$ such that*

$$A = 2cd, \quad B = i(ad - bc), \quad C = 2ab, \quad D = ad + bc.$$

Proof. We begin by showing there is a bijection

$$\mathbb{P}^1(\mathbb{Q}(i)) \times \mathbb{P}^1(\mathbb{Q}(i)) \rightarrow X = \{[A, B, C, D] : AC = B^2 + D^2\} \subset \mathbb{P}^3(\mathbb{Q}(i))$$

given by

$$[a, c] \times [b, d] \mapsto [2cd, i(ad - bc), 2ab, ad + bc]$$

A brief calculation verifies that the image of the map falls in X . An inverse map is given by

$$[A, B, C, D] \mapsto [C, Bi - D] \times [C, -Bi - D] = [-Bi - D, A] \times [Bi - D, A].$$

(At least one of these is always defined, and they are equal when both defined since $AC = B^2 + D^2$.)

Now suppose one has $A, B, C, D \in \mathbb{Z}[i]$ where A and C are divisible by 2, and $AC = B^2 + D^2$. Let $G = Bi - D$ and let $G' = -Bi - D$. Since $AC = GG'$, we can find Gaussian integers A_1, A_2, B_1 and B_2 such that

$$G = A_1C_1, \quad G' = A_2C_2, \quad A = A_1A_2, \quad C = C_1C_2.$$

Then taking

$$a = \frac{C}{C_1} = C_2, \quad c = \frac{G}{C_1} = A_1, \quad b = \frac{C}{C_2} = C_1, \quad d = \frac{G'}{C_2} = A_2.$$

gives

$$(2cd, i(ad - bc), 2ab, ad + bc) = (2A, 2B, 2C, 2D).$$

Since $2 = (1 + i)(1 - i)$ divides A and C , it divides cd and ab . We also have that ad and bc are congruent modulo 2. Let $\pi = 1 + i$. It must be, therefore, that π divides a and c or else b and d . In the former case, we may replace a with a/π and c with c/π ; this now gives

$$(2cd, i(ad - bc), 2ab, ad + bc) = ((1 - i)A, (1 - i)B, (1 - i)C, (1 - i)D).$$

The latter case is similar. Repeating the argument with $\pi' = 1 - i$, we obtain the necessary a, b, c, d . \square

Lemma 12.6. *If $z_1, z_2, z_3, z_4 \in \mathbb{Z}[i]$ are centre-curvatures of a primitive Descartes configuration in \mathcal{S}_1 , then $z_1 \equiv z_2 \equiv z_3 \equiv z_4 \pmod{2}$.*

Proof. By the last statement of Proposition 11.1, it suffices to show that whenever two distinct circles in \mathcal{S}_1 are tangent, they are in bijection with elements of the same coset of $\text{PSL}_2(\mathbb{Z}[i]) / \text{PSL}_2(\mathbb{Z})$ in $\text{PGL}_2(\mathbb{Z}[i]) / \text{PSL}_2(\mathbb{Z})$. Without loss of generality, we can assume one of the circles is the real line. Then it suffices to show that any image of \mathbb{R} under $M \in \text{PGL}_2(\mathbb{Z}[i])$ which is tangent to \mathbb{R} satisfies $M \in \text{PSL}_2(\mathbb{Z}[i])$. Since the point of tangency corresponds to a primitive vector of $\mathbb{Z}[i]^2$ with coordinates in \mathbb{Z} , we can also assume without loss of generality, by pre-composing with an element of $\text{PSL}_2(\mathbb{Z})$, that the point of tangency is fixed by M . Conjugating M by an element of $\text{PSL}_2(\mathbb{Z})$, we may assume the point of tangency is the origin. Then M has the form

$$\begin{pmatrix} \alpha & 0 \\ \beta & 1 \end{pmatrix}.$$

Since M has determinant α , it must be that α is a unit. By precomposing with an element of $\text{PSL}_2(\mathbb{Z})$, we can make β purely imaginary. Since α/β is not on the real line, it must be that α is ± 1 , and we are done. \square

A criterion for four circles to form a Descartes configuration has already been described in [12, Theorem 3.1] and [16, Theorem 2.2], and it subsumes the usual Descartes relation. We cite it here, but in another form (under a change of basis). Let the product of the centre and curvature of a circle be called the *centre-curvature*.

Theorem 12.7 (Theorem 3.1 of [12]). *Let*

$$\mathbf{R} = \frac{1}{2} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad \mathbf{S} = \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 & 2 \\ 1 & 1 & -1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Suppose we have four circles of oriented curvatures (b_1, b_2, b_3, b_4) and centre-curvatures (z_1, z_2, z_3, z_4) . Define the curvature-centre-conjugate matrix

$$\mathbf{M} = \begin{pmatrix} b_1 & z_1 & \overline{z_1} \\ b_2 & z_2 & \overline{z_2} \\ b_3 & z_3 & \overline{z_3} \\ b_4 & z_4 & \overline{z_4} \end{pmatrix}.$$

If the circles form a Descartes configuration, then \mathbf{M} has nonzero first column and satisfies

$$\mathbf{M}^T \mathbf{S}^T \mathbf{R} \mathbf{S} \mathbf{M} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}. \quad (11)$$

Conversely, any solution \mathbf{M} to (11) with nonzero first column, and conjugate second and third columns is the curvature-centre-conjugate matrix of a unique ordered, oriented Descartes configuration.

Corollary 12.8. *If $z_1, z_2, z_3, z_4 \in \mathbb{Z}[i]$ are the centre-curvatures of four circles in a primitive Descartes configuration, then there are generically two possible choices for the collection of curvatures b_1, b_2, b_3, b_4 . These solutions correspond to two Descartes quadruples of circles which are carried to one another by the map $z \mapsto -2/\bar{z}$.*

Proof. By assumption, there is at least one primitive Descartes configuration having given centre-curvatures, hence at least one possible collection of b_i . If the z_i are known, the b_i must satisfy the three conditions corresponding to the entries in the first row of the matrix in (11). The matrix

$$\mathbf{S}^T \mathbf{R} \mathbf{S} = \frac{1}{4} \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix},$$

is full rank. The vectors (z_1, z_2, z_3, z_4) and $(\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4)$ are independent since $(\operatorname{Re}(z_i))_{i=1}^4$ and $(\operatorname{Im}(z_i))_{i=1}^4$ are (otherwise the quadruple would consist of circles with collinear centres). Therefore the two homogeneous linear conditions are independent. Hence there is a two-parameter family of solutions which must satisfy a homogeneous quadratic condition. The condition that the b_i are integers with no common factor determines the two solutions uniquely.

Furthermore, if one applies the map $z \mapsto -2/\bar{z}$ to a Descartes quadruple of circles, then one obtains another Descartes quadruple. By Proposition 11.1, the centre-curvatures do not change. If the two quadruples are distinct, the curvatures must change (since the centre-curvatures and curvatures together determine the quadruple). \square

Proof of Theorem 12.1. Proposition 12.3 shows that all possible Descartes quadruples (and hence Apollonian circle packings) occur in \mathcal{S}_1 , up to rigid motions. It also shows that to each chamber is associated a primitive Descartes quadruple. It remains to show the converse.

Since by Proposition 11.1 all circles in \mathcal{S}_1 are strongly integral, let $z_1, z_2, z_3, z_4 \in \mathbb{Z}[i]$ represent the centre-curvatures of four circles of \mathcal{S}_1 in Descartes configuration, and let $b_1, b_2, b_3, b_4 \in \mathbb{Z}$ represent the oriented curvatures, respectively. Let

$$A = z_1 + z_4, \quad B = \frac{z_1 + z_2 - z_3 + z_4}{2}, \quad C = z_2 + z_4, \quad D = z_4.$$

By Theorem 12.7, $B^2 + D^2 = AC$ and B is a Gaussian integer. Furthermore, the numbers $\operatorname{Re}(z_i)$ must all be congruent modulo 2, as must the numbers $\operatorname{Im}(z_i)$, by Lemma 12.6. Hence $A/2$ and $C/2$ are Gaussian integers.

We are in the case of Lemma 12.5, and there are some $\alpha, \beta, \gamma, \delta \in \mathbb{Z}[i]$ such that

$$2\alpha\beta = A, \quad i(\alpha\delta - \gamma\beta) = B, \quad 2\gamma\delta = C, \quad \alpha\delta + \gamma\beta = D. \quad (12)$$

Set

$$u = \begin{pmatrix} \alpha \\ \bar{\beta} \end{pmatrix}, \quad v = \begin{pmatrix} i\gamma \\ i\bar{\delta} \end{pmatrix}, \quad w = -u - v.$$

Let $Z \begin{pmatrix} \epsilon & \eta \\ \zeta & \theta \end{pmatrix} = i(\eta\bar{\zeta} - \epsilon\bar{\theta})$ give the centre-curvature of the circle which is the image of \mathbb{R} under $\begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \epsilon & \eta \\ \zeta & \theta \end{pmatrix}$, as in Proposition 11.1. One checks that

$$z_1 = 2\alpha\beta - \alpha\delta - \gamma\beta = Z(v, iw), \quad (13)$$

$$z_2 = 2\gamma\delta - \alpha\delta - \gamma\beta = Z(w, iu), \quad (14)$$

$$z_3 = 2\gamma\delta + 2\alpha\beta - 2i\gamma\beta + 2i\alpha\delta - \alpha\delta - \gamma\beta = Z(w - iv, iu - v), \quad (15)$$

$$z_4 = \alpha\delta + \gamma\beta = Z(u, iv). \quad (16)$$

Also a consequence of Theorem 12.7 is that

$$B\bar{B} + D\bar{D} - \frac{1}{2}A\bar{C} - \frac{1}{2}\bar{A}C = 2.$$

However, we have found expressions for A, B, C, D in terms of $\alpha, \beta, \gamma, \delta$; plugging these in, we obtain

$$B\bar{B} + D\bar{D} - \frac{1}{2}A\bar{C} - \frac{1}{2}\bar{A}C = 2N_{\mathbb{Q}(i)/\mathbb{Q}}(\alpha\bar{\delta} - \gamma\bar{\beta}),$$

where $N_{\mathbb{Q}(i)/\mathbb{Q}}$ is the Gaussian norm. This implies that $\alpha\bar{\delta} - \gamma\bar{\beta}$ is a unit. Therefore, u and v form a $\mathbb{Z}[i]$ -basis of $\mathbb{Z}[i]^2$, and the values z_1, z_2, z_3, z_4 are the values of Z at the corners of one chamber of the Apollonian city, i.e. we have found a Descartes quadruple with the required centre-curvatures.

If we replace $(\alpha, \beta, \gamma, \delta)$ with $(\delta, \gamma, \beta, \alpha)$ in (12), then we recover the values (C, B, A, D) instead of (A, B, C, D) ; this corresponds to the same set of centre-curvatures in a new order z_2, z_1, z_3, z_4 . Then we define instead

$$u = \begin{pmatrix} \delta \\ \bar{\gamma} \end{pmatrix}, \quad v = \begin{pmatrix} i\beta \\ i\bar{\alpha} \end{pmatrix}, \quad w = -u - v.$$

Then instead of (13)–(16) we have

$$z_2 = 2\gamma\delta - \alpha\delta - \gamma\beta = Z(v, iw),$$

$$z_1 = 2\alpha\beta - \alpha\delta - \gamma\beta = Z(w, iu),$$

$$z_3 = 2\gamma\delta + 2\alpha\beta - 2i\gamma\beta + 2i\alpha\delta - \alpha\delta - \gamma\beta = Z(w - iv, iu - v),$$

$$z_4 = \alpha\delta + \gamma\beta = Z(u, iv).$$

As before, u and v form a $\mathbb{Z}[i]$ -basis of $\mathbb{Z}[i]^2$. This represents a second chamber with the same centre-curvatures, but generically different curvatures. These two different chambers correspond to quadruples which are related by applying the map $z \mapsto -2/\bar{z}$ on \mathbb{C} .

Corollary 12.8 implies that these two primitive Descartes quadruples correspond to the two possible quadruples with given centre-curvatures. Therefore, one of the two chambers we've found is exactly the quadruple we began with.

An Apollonian circle packing is formed by completing all triples of circles, beginning with a specific originating quadruple; correspondingly, each of the completing circles is obtained as the vertex of an adjacent chamber. Hence all the circles in the packing correspond to vertices in the same palace as the originating chamber, and all the vertices in the palace give rise to circles in the Apollonian circle packing. \square

13. APOLLONIAN CIRCLE PACKINGS

Consider the group

$$\Gamma' = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}^{-1} \Gamma \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$

This group acts by right multiplication on lax vertex lattice matrices. If we consider the palace containing the lax vertex lattice $(1, 0), (0, 1)$ corresponding to the real line, then Γ' takes a lax vertex lattice in this palace to all other lax vertex lattices in the *same palace* (by the proof of Proposition 9.1).

Hence the following is a corollary to Proposition 9.1 and Theorem 12.1.

Corollary 13.1. *The collection of images of the real line under the Möbius transformations $\begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} N \Gamma'$ for some $N \in \text{PGL}_2(\mathbb{Z}[i])$ form a primitive integral Apollonian circle packing.*

Figure 7, which is obtained as all images of the real line under Γ' , shows all the lax vertex lattices in the palace containing the lax vertex lattice $(1, 0), (0, 1)$. We obtain the horizontal Apollonian strip packing (it fills an infinite horizontal strip with this repeating pattern; the horizontal lines are circles of zero curvature).

The curvatures of all the circles shown in this packing are even, which explains the convention of dilating⁶ these pictures by a factor of 2 in Theorem 8.1. Hence, we obtain the more usual strip packing (with an infinite string of unit circles) by taking all images of \mathbb{R} under $\begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \Gamma'$.

⁶This is the lesser of several evils, each of which results in a proliferation of terrible twos.

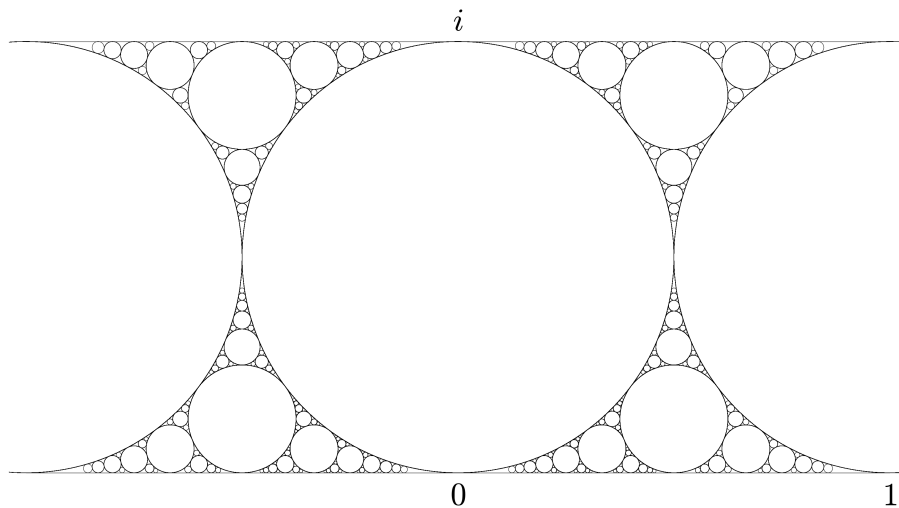


FIGURE 7. The Apollonian strip packing, obtained as all images of \mathbb{R} under Γ' .

If we use instead a left coset $N\Gamma'$ of Γ' , for some $N \in \mathrm{PGL}_2(\mathbb{Z}[i])$, we obtain other Apollonian circle packings. Figure 8 is obtained upon using $N = \begin{pmatrix} i & 0 \\ 1 & i \end{pmatrix}$.

If we take together all the left cosets $\begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} N\Gamma'$ of Γ' for $N \in \mathrm{PSL}_2(\mathbb{Z}[i])$, drawing all the images of the real line, we obtain an Apollonian superpacking, as studied by Graham, Lagarias, Mallows, Wilks and Yan [13]. This picture, which those authors call the *standard strongly integral super-packing*, is shown in red in Figure 2. This repeating pattern contains one copy of every Apollonian circle packing in its unit square (except the strip packing). For details, see [13].

From the explicit bijection between lax circle lattices and certain equivalence classes of binary quadratic forms which we will see in Section 14, it will be apparent that a class number counting quadratic forms of given discriminant can be used to count the number of distinct occurrences of a given curvature in a region of the super-packing. This was already studied in an earlier paper by Graham, Lagarias, Mallows, Wilks and Yan [11, Theorem 4.2].

The Apollonian superpacking has a dual, which consists of the images of the real line under all $\begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} N\Gamma'$ for $N \in \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \mathrm{PSL}_2(\mathbb{Z}[i])$, and is shown in blue in Figure 2. The superpacking together with its dual, consist of all images of the real line under $\begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \mathrm{PGL}_2(\mathbb{Z}[i])$. The resulting riot of circles can be considered a picture of the collection of full

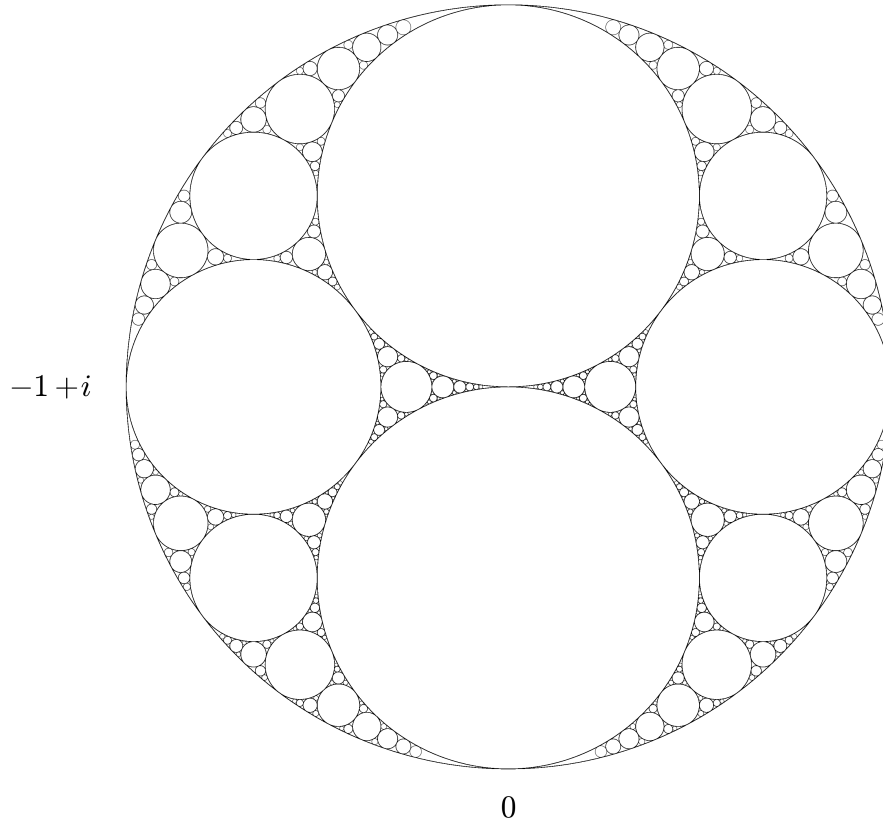


FIGURE 8. The Apollonian $(-1, 2, 2, 3)$ packing, obtained as all images of \mathbb{R} under $\begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} i & 0 \\ 1 & i \end{pmatrix} \Gamma'$.

rank sublattices of \mathbb{Z}^2 (modulo lax equivalence), as described in the next section, or, simply, a picture of $\mathrm{PGL}_2(\mathbb{Z}[i])/\mathrm{PSL}_2(\mathbb{Z})$.

14. THE VIEW FROM A VERTEX: CONWAY'S SENSUAL QUADRATIC FORM

The Apollonian city is not only formed in analogy to Conway's topograph, but it in fact *includes* infinitely many copies of the topograph. The topograph can be highlighted by choosing any one vertex of the Apollonian city and restricting attention only to those vertices at distance one from the chosen vertex.

Theorem 14.1. *Fix a vertex of the Apollonian city with vertex lattice L . Then the adjacent vertices may be put in bijection with the primitive elements $x \in L$ in such a way that the following hold.*

- (1) *Two such vertices are connected by an edge of the Apollonian city if and only if their corresponding $x, x' \in L$ form a basis for L .*

- (2) Let H be the imaginary part of a Hermitian form. Then the vertices have H -values

$$H(ix, x) - H(L).$$

Furthermore, $H(ix, x)$ is a binary quadratic form on the \mathbb{Z} -module L of discriminant

$$4 \left(H(u, v)^2 - H(iu, u)H(iv, v) \right),$$

where u, iv is any ordered basis of L .

Proof. Fix any one vertex of the Apollonian city, which we will call the *mother vertex*. Suppose it is labelled with the vertex lattice L generated by u and iv , and suppose that we have labelled this palace with one of the two possible global split labellings, as in the proof of Proposition 3.2. Now survey the *motherland*: those vertices at distance one from the mother, which we will call *subjects*, together with the edges which connect them.

Fix a subject. Consider the split labelling on the edge joining the mother to the subject. The label closest to the mother is a vector x which is in the mother's vertex lattice. Thus we associate each subject with this corresponding x .

Two vertices in the motherland are joined by an edge if and only if, together with the mother, they form a wall in the Apollonian city. This is the case if and only if together, they form a \mathbb{Z} -basis for L (see the proof of Proposition 3.2).

The function

$$g : \mathbb{Z}[i]^2 \rightarrow \mathbb{Z}, \quad g(x) = H(ix, x) = h(x, x),$$

satisfies the parallelogram law $g(x + y) + g(x - y) = 2(g(x) + g(y))$. Restricting g to L , we obtain a binary quadratic form. It takes values on the collection of edges connecting the mother with her subjects, so it can be viewed as a form on the motherland. The values it takes are exactly the H -labels of those edges in the Apollonian city. The discriminant of the form can be calculated as

$$4 \left(H(u, v)^2 - H(iu, u)H(iv, v) \right).$$

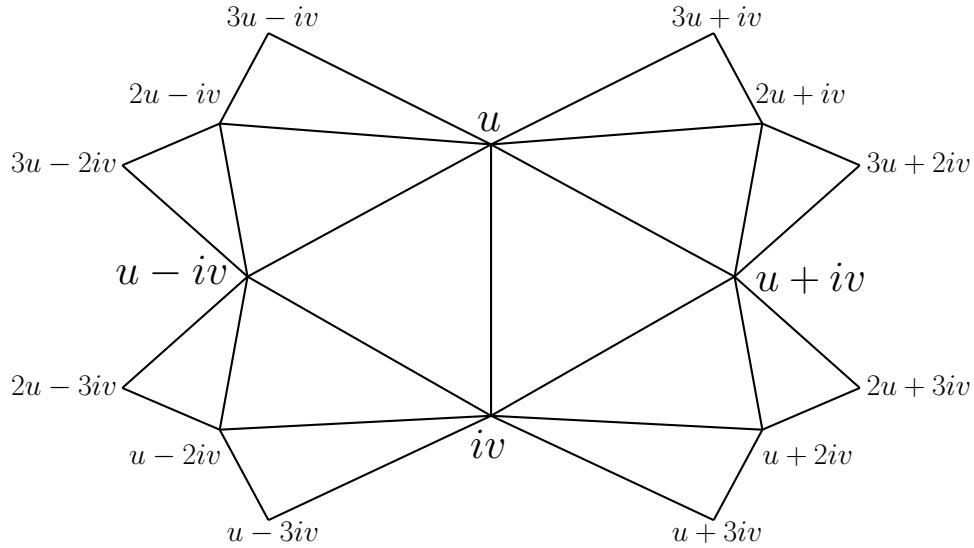
This association of a subject in the motherland to a primitive vector $x \in L$ implies that there exists some basis x, iy of L . Let z be such that $x + y + z = 0$. Then the subject has vertex lattice generated by z, ix . Property (H2) of H implies that

$$H(z, ix) = g(x) - C$$

where the constant $C = H(x, iy)$ is the H -label of the mother. In other words, the H -labelling of the collection of vertices in the motherland is a translated quadratic form.

Finally, note that the definition of the quadratic form is actually only dependent on the *lax* vertex lattice. \square

With reference to the language of the proof, suppose we illustrate just those edges of the Apollonian city which connect vertices in the motherland, and label the vertices by the vector of L associated to them. Then we obtain something very familiar (compare the first picture in Section 5):



In other words, the motherland is a copy of Conway's topograph (in its 'dual' triangular form).

15. CIRCLE LATTICES

The curvature form

$$B((\alpha, \beta), (\gamma, \delta)) = \text{Im}(\beta\bar{\delta}),$$

of Proposition 11.1 depends only on the second coordinates of its arguments. Considering $\beta = a + bi$, $\delta = c + di \in \mathbb{Z}[i]$ as vectors (a, b) , $(c, d) \in \mathbb{Z}^2$, we can think of each vertex of the Apollonian city as being labelled by an oriented full-rank sublattice of \mathbb{Z}^2 generated by (a, b) and (c, d) , instead of a vertex lattice generated by (α, β) and (γ, δ) . This is a *circle lattice*, as in Definition 4.3. The corresponding *lax circle lattice* is an equivalence class of two circle lattices:

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}, \quad \begin{pmatrix} b & d \\ -a & -c \end{pmatrix}.$$

Proposition 15.1. *Given a circle in \mathcal{S}_1 , there is a unique lax circle lattice that may be associated to it. Furthermore, the circle has oriented curvature equal to the negative of the determinant of the circle lattice.*

A circle lattice determines the oriented curvatures of the Descartes quadruples and the Apollonian circle packing in \mathcal{S}_1 containing any circle to which it is associated.

Proof. The first statement is a consequence of Theorem 8.1, since the circle lattice is determined by the vertex lattice. The circle lattices $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and $\begin{pmatrix} b & d \\ -a & -c \end{pmatrix}$ both have determinant

$$ad - bc = -B((\alpha, \beta), (\gamma, \delta)),$$

which is the negative of the curvature of the circle associated it.

For the last statement, the bases of the circle lattice are the second coordinates of the bases u and iv of the associated vertex lattice. This, in turn, determines the curvatures of any Descartes quadruple associated to a chamber containing that vertex, by the proof of Proposition 12.3. In particular, if the second coordinates of u and v are $a + bi$ and $d - ci$, respectively (corresponding to the form of the circle lattice above), then for the first standard ultrabasis chamber associated to superbasis $u, v, w = -u - v$, these are

$$\begin{aligned} b_1 &= -(ad - bc) = B(u, iv), \\ b_2 &= c^2 + d^2 + (ad - bc) = B(w, iu), \\ b_3 &= (a + c)^2 + (b + d)^2 + (ad - bc) = B(v, iw), \\ b_4 &= a^2 + b^2 + (ad - bc) = B(w - iv, iu - v). \end{aligned}$$

Since Descartes quadruples in \mathcal{S}_1 are in bijection with chambers (Theorem 12.1), this determines the curvatures of any Descartes quadruple containing a circle with that circle lattice. The curvatures of one Descartes quadruple determine the curvatures of the entire Apollonian circle packing of which it is a member. \square

A given lax circle lattice may occur more than once in an Apollonian circle packing, but only if a given Descartes quadruple occurs repeatedly: the only example in which it may occur infinitely often is the strip packing, where, for example, each of the infinitely many circles of radius one necessarily has circle lattice \mathbb{Z}^2 . All other Apollonian circle packings contain at most finitely many circles of curvature below a given bound (see [5, 11] for counting estimates).

The packing shown in Figure 8 has a symmetry, and therefore the same lax circle lattice often occurs twice or four times. In Figure 4, we show a circle lattice associated to each circle in that packing.

It is nice to record the resulting *lattice Descartes rule*, which gives, for any three circle lattices which appear in an Apollonian circle packing, the circle lattices attached to the two circles completing the triple to a Descartes quadruple.

Theorem 15.2. *Consider three mutually tangent circles in an Apollonian circle packing. For some a, b, c, d , they have the following circle lattices*

(represented by matrices whose column vectors form a basis):

$$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \quad B = \begin{pmatrix} d & b-c \\ -c & -a-d \end{pmatrix}, \quad C = \begin{pmatrix} -a-d & -b \\ -b+c & a \end{pmatrix}.$$

The two completing circles have the following circle lattices:

$$D_1 = \begin{pmatrix} -b-d & c+a+d \\ a+c & d+b-c \end{pmatrix}, \quad D_2 = \begin{pmatrix} c-a & d-b+c \\ d-b & -c+a+d \end{pmatrix}.$$

Proof. By Theorem 12.1, the triple of circles is associated to a superbasis in the Apollonian city. Choose vectors u, v, w representing the superbasis such that $u + v + w = 0$, choose a global split labelling (as in the proof of Proposition 3.2), and hence a labelling by non-lax vertex lattices, where one vertex has vertex lattice generated by u, iv . For the first coordinates of u and iv , write $a + bi$ and $c + di$ respectively. The result follows. \square

In practical application, to find the matrix forms for A, B, C above, so that one may apply this theorem, one may use the proofs of Proposition 12.3 and Lemma 12.4.

16. QUADRATIC FORMS IN CURVATURES

Besides a circle lattice, each circle also has an associated quadratic form, as described in Section 14. In this section we consider the quadratic form obtained in Theorem 14.1 by using the curvature form B as in the last section.

Graham, Lagarias, Mallows, Wilks and Yan have already described a quadratic form associated to a root quadruple of an Apollonian circle packing in [11, Theorem 4.2]; their form lies in the equivalence class associated to the ‘outer’ circle of a packing in what follows. Bourgain and Fuchs have also described the same form in terms of a quadruple of curvatures [3]. These are all essentially the same form.

For a circle associated to a vertex lattice with basis u, iv having second coordinates β and δ respectively, this quadratic form obtained from the curvature Hermitian form is

$$g(Xu + Yiv) = N_{\mathbb{Q}(i)/\mathbb{Q}}(X\beta + Y\delta).$$

Using the convention that $\beta = a + bi$ and $\delta = c + di$, the discriminant of this form is related to the curvature of the circle:

$$-4H((\alpha, \beta), (\gamma, \delta))^2 = -4(ad - bc)^2,$$

A representative of the equivalence class is given by

$$\begin{aligned} & \beta\bar{\beta}X^2 + 2\operatorname{Re}(\beta\bar{\delta})XY + \delta\bar{\delta}Y^2 \\ &= (a^2 + b^2)X^2 + 2(ac + bd)XY + (c^2 + d^2)Y^2. \end{aligned}$$

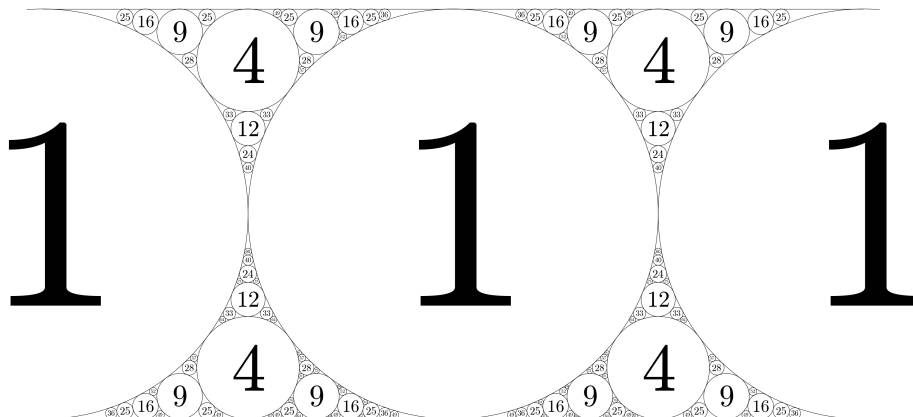


FIGURE 9. The Apollonian strip packing, with curvatures.

The matrix form Q of this quadratic form is related to the matrix M of the circle lattice (whose column vectors are the first coordinates of u and iv). In particular, $M^T M = Q$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{pmatrix}.$$

Note that the resulting quadratic form is a primitive form: $\gcd(a^2 + b^2, ac + bd, c^2 + d^2) = 1$ (otherwise $a + bi$ and $c + di$ are not coprime).

The circles which are tangent to a fixed circle of lattice β, δ form *the motherland*, as in Section 14. They have curvatures

$$N_{\mathbb{Q}(i)/\mathbb{Q}}(X\beta + Y\delta) - ad + bc.$$

For example, consider the unit circle in the strip packing (Figure 7, drawn dilated by a factor of two). It has circle lattice $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The circles tangent to it have values $X^2 + Y^2 - 1$, for all primitive vectors $(X, Y) \in \mathbb{Z}^2$. For example, one sees n^2 for all $n \in \mathbb{Z}$ but $n^2 - 1$ only when $n = 1$ or n is the largest member of a pythagorean triple. These values are

$$0, 1, 4, 9, 12, 16, 24, 25, 28, 33, 36, 40, \dots$$

See Figure 9.

Proposition 16.1. *If matrix*

$$Q = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

represents a primitive positive definite quadratic form, whose discriminant is $-4n^2$ for some $n \in \mathbb{Z}$, then

$$Q = M^T M$$

for some integral matrix M of determinant $\pm n$. Furthermore, there are exactly four solutions for M , of the form

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}, \begin{pmatrix} -a & -c \\ -b & -d \end{pmatrix}, \begin{pmatrix} b & d \\ -a & -c \end{pmatrix}, \begin{pmatrix} -b & -d \\ a & c \end{pmatrix}.$$

Conversely, if

$$M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

is any integral lattice, then $M^T M$ is the matrix of a primitive positive definite quadratic form with discriminant $-4(ad - bc)^2$.

Proof of Proposition 16.1. The existence of M amounts to solving the equations

$$A = a^2 + b^2, B = ac + bd, C = c^2 + d^2, D = ad - bc, AC - B^2 = D^2.$$

Since the discriminant is negative, $AC > 0$. But since the form is positive definite, $A, C > 0$. From primitivity, A, B, C, D have no common factor. The result follows from Lemma 12.4.

The converse is immediate. A form with negative discriminant but $A, C > 0$ is positive definite. \square

Hence a primitive quadratic form of the type of Proposition 16.1 arises from some circle lattice, and is part of a Descartes quadruple of some packing. One can recover a Descartes quadruple simply by evaluating the quadratic form on a superbasis.

Corollary 16.2. *We have an explicitly computable bijection:*

$$\begin{array}{c} \{\text{lax circle lattice}\} \\ \updownarrow \\ \left\{ \begin{array}{l} \text{equivalence class of primitive positive definite} \\ \text{integral binary quadratic forms} \\ \text{of discriminant negative four times a square} \end{array} \right\} \end{array}$$

The lattice Descartes rule of Theorem 15.2 gives rise to a corresponding quadratic form Descartes rule.

17. APPENDIX: BESTVINA AND SAVIN'S TOPOGRAPH

Bestvina and Savin [2] study a complex formed of bases and superbases in $\mathbb{Z}[i]^2$. Because a basis is contained in four superbases, each basis is associated to a square 2-cell, whose edges are the superbases. Square 2-cells are glued three along each edge, reflecting the fact that a superbasis contains three bases. The link of a vertex is the 1-skeleton of a cube, which describes which superbases share common bases.

To compare Bestvina and Savin's complex to the Apollonian city studied here, consider the six faces of the cubical link. These correspond to vectors

$u, v, w, u + iv, v + iw, w + iu$, the edges of a tetrahedron in the Apollonian city. The eight vertices of the cube can be broken up into two collections of four, each of which forms the vertices of a tetrahedron: the collection of diagonals of each face which connect the four vertices gives the edges of the tetrahedron.

In this way, each vertex of Bestvina and Savin's complex contains two dual tetrahedra. As mentioned briefly in Section 3, an ultrabasis tetrahedron has a natural dual, since the vectors labelling the edges adjacent to a vertex form a superbasis. This corresponds to the dual Descartes quadruple described by Coxeter [9].

There are other pieces of the vector/basis/superbasis/ultrabasis description of $\mathbb{P}^1(\mathbb{Q}(i))$ that remain to be studied; these complexes are only some of the possible complexes formed by the relationships between these objects.

18. FINAL COMMENTS

A future paper will study the relationships between the reduction of quadratic forms, continued fractions in the Gaussian integers, and reduction in Apollonian circle packings. Other natural questions include the study of other quadratic number fields (notably the field with unit group of order six), as well as a wide range of questions concerning the arrangement of lattices into circle packings.

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